Chapter 2

Exercise 2A

There is no need for worked solutions for any of the questions in this exercise. Refer to the answers in Sadler.

Exercise 2B

1. (a) Amplitude of $\sin(x)$ is 1.
   (b) Amplitude of $\cos(x)$ is 1, so amplitude of $2\cos(x)$ is 2.
   (c) Amplitude of $\cos(x)$ is 1, so amplitude of $4\cos(x)$ is 4.
   (d) Amplitude of $\sin(x)$ is 1, so amplitude of $-3\sin(2x)$ is 3. (Remember, amplitude can't be negative. The 2 here affects the period, not the amplitude.)
   (e) Amplitude of $\cos(x)$ is 1, so amplitude of $2\cos\left(x+\frac{\pi}{2}\right)$ is 2. (The $+\frac{\pi}{2}$ here affects the phase position, not the amplitude.)
   (f) Amplitude of $\sin(x)$ is 1, so amplitude of $-3\sin(x-\pi)$ is 3. (Remember, amplitude can't be negative. The $-\pi$ here affects the phase position, not the amplitude.)
   (g) Amplitude of $\cos(x)$ is 1, so amplitude of $5\cos(x-2)$ is 5. (The -2 here affects the phase position, not the amplitude.)
   (h) Amplitude of $\cos(x)$ is 1, so amplitude of $-3\cos(2x+\pi)$ is 3. (Amplitude can't be negative; the 2 affects period, not amplitude and the $+\pi$ affects the phase position, not the amplitude.)

2. (a) Period of $\sin x$ is $360°$.
   (b) Period of $\tan x$ is $180°$.
   (c) Period of $\sin x$ is $360°$ so the period of $2\sin x$ is also $360°$. (The 2 affects amplitude, not period.)
   (d) Period of $\sin x$ is $360°$ so the period of $\sin 2x$ is $\frac{360}{2} = 180°$.
   (e) Period of $\cos x$ is $360°$ so the period of $\cos\frac{x}{2}$ is $\frac{360}{\frac{1}{2}} = 720°$.
   (f) Period of $\cos x$ is $360°$ so the period of $\cos 3x$ is $\frac{360}{3} = 120°$.
   (g) Period of $\tan x$ is $180°$ so the period of $3\tan 2x$ is $\frac{180}{2} = 90°$. (The 3 does not affect the period.)
   (h) Period of $\sin x$ is $360°$ so the period of $3\sin\frac{x-60°}{3}$ is $\frac{360}{\frac{1}{3}} = 1080°$. (The first 3 affects amplitude, not period. The $-60°$ affects phase position, not period.

3. (a) Period of $\cos x$ is 2π.
   (b) Period of $\tan x$ is π.
   (c) Period of $\cos x$ is 2π so the period of $3\cos x$ is 2π. (The 3 affects amplitude, not period.)
   (d) Period of $\cos x$ is 2π so the period of $2\cos 4x$ is $\frac{2\pi}{4} = \frac{\pi}{2}$. (The 2 affects amplitude, not period.)
   (e) The period of $\tan x$ is π so the period of $2\tan 3x$ is $\frac{\pi}{3}$. (The 2 does not affect period.)
   (f) The period of $\sin x$ is 2π so the period of $\frac{1}{2}\sin 3x$ is $\frac{2\pi}{3}$. (The $\frac{1}{2}$ affects amplitude, not period.)
   (g) The period of $\sin x$ is 2π so the period of $3\sin\frac{x}{2}$ is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. (The 3 affects amplitude, not period.)
   (h) The period of $2\cos 4x$ is $\frac{2\pi}{4} = \frac{\pi}{2}$. (The 2 affects amplitude, not period.)
   (i) Period of $\cos x$ is 2π so the period of $2\cos(2x-\pi)$ is $\frac{2\pi}{2} = \pi$. (The first 2 affects amplitude, not period; the $-\pi$ affects phase position, not period.
   (j) The period of $\sin x$ is 2π so the period of $2\sin 4\pi x$ is $\frac{2\pi}{4\pi} = \frac{1}{2}$. (The 2 affects amplitude, not period.)

4. (a) The maximum of $\sin x$ is 1 and occurs when $x = \frac{\pi}{2}$: coordinates $\left(\frac{\pi}{2}, 1\right)$
   The minimum of $\sin x$ is -1 and occurs when $x = \frac{3\pi}{2}$: coordinates $\left(\frac{3\pi}{2}, -1\right)$
   (b) The "2+" increases both maximum and minimum by 2 and has no effect on when they occur. Maximum at $\left(\frac{\pi}{2}, 3\right)$; minimum at $\left(\frac{3\pi}{2}, 1\right)$.
   (c) The "$-$" has the effect of reflecting the graph of $\sin(x)$ in the $x-$axis, so the maximum becomes the minimum and vice versa. Maximum at $\left(\frac{3\pi}{2}, 1\right)$; minimum at $\left(\frac{\pi}{2}, -1\right)$.

(d) The "2" decreases the period from 2π to π. The $x-$position of maximum and minimum is similarly halved to $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ respectively. In addition, the decreased period means that we will get two full cycles in the domain $0 \le x \le 2\pi$ so there will be two maxima and two minima, each separated by π. The "+3" means the maxima will have $y-$ values of $1+3=4$ and minima of $-1+3=2$. Thus, maxima at $\left(\frac{\pi}{4},4\right)$ and $\left(\frac{5\pi}{4},4\right)$ and minima at $\left(\frac{3\pi}{4},2\right)$ and $\left(\frac{7\pi}{4},2\right)$.

(e) The "$-\frac{\pi}{4}$" moves the graph of $\sin x$ to the right $\frac{\pi}{4}$ units so the $x-$coordinate of maximum and minimum increase to $\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}$ and $\frac{3\pi}{2}+\frac{\pi}{4}=\frac{7\pi}{4}$. The +3 increases maximum and minimum to 4 and 2. Thus, maximum $\left(\frac{3\pi}{4},4\right)$, minimum $\left(\frac{7\pi}{4},2\right)$.

5. (a) The maximum value of $\sin x$ is 1 so the maximum value of $3\sin x$ is $3\times 1=3$. The smallest positive value of x that gives this maximum is 90°.
   (b) The maximum value is 2 when $x=90+30=120°$.
   (c) The maximum value is 2 when $x=90-30=60°$.
   (d) The maximum value is 3 when $x=270°$.

6. (a) The maximum value is 3 when $x=\frac{\pi}{2}\div 2=\frac{\pi}{4}$.
   (b) The maximum value is 5 when $x=\frac{3\pi}{2}$.
   (c) The maximum value is 2. The maximum of $\cos x$ occurs when $x=0$ so here the maximum occurs when $x=0-\frac{\pi}{6}=-\frac{\pi}{6}$, but this is not positive so we must add the period to get $x=-\frac{\pi}{6}+2\pi=\frac{11\pi}{6}$
   (d) The maximum value is 3 when $x=0+\frac{\pi}{6}=\frac{\pi}{6}$

7. (a) Amplitude is 2, curve is not reflected, so $a=2$.
   (b) Amplitude is 3, curve is not reflected, so $a=3$.
   (c) Amplitude is 3, curve is reflected, so $a=-3$.
   (d) Amplitude is about 1.4, curve is reflected, so $a=-1.4$.

8. (a) Amplitude is 3, curve is not reflected, so $a=3$.
   (b) Amplitude is 2, curve is reflected, so $a=-2$.

9. (a) At $x=\frac{\pi}{4}$ the $y-$value is 2, so $a=2$.
   (b) At $x=45°$ the $y-$value is -1, so $a=-1$.

10. (a) Amplitude is 2, so $a=2$. Period is $\frac{2\pi}{3}$ which is one third the period of $\sin x$ so $b=3$.
    (b) Amplitude is 3 and curve is reflected, so $a=-3$. Period is π which is half the period of $\sin x$ so $b=2$.
    (c) Amplitude is 2, so $a=2$. Period is 60° which is one sixth the period of $\sin x$ so $b=6$.
    (d) Amplitude is 3, so $a=3$. Period is 90° which is one quarter the period of $\sin x$ so $b=4$.

11. (a) Amplitude is 1 so $a=1$. Period is π which is half the period of $\cos x$ so $b=2$.
    (b) Amplitude is 3 and the curve is reflected, so $a=-3$. Period is $\frac{2\pi}{3}$ which is one third the period of $\cos x$ so $b=3$.
    (c) Amplitude is 3 and the curve is reflected, so $a=-3$. Period is 180° which is half the period of $\cos x$ so $b=2$.
    (d) Amplitude is 2 so $a=2$. Period is 90° which is quarter the period of $\cos x$ so $b=4$.

12. (a) Amplitude is 2, so $a=2$. The broken line is $y=2\sin x$. The unbroken line is shifted to the right by 30° so its equation is $y=2\sin(x-30°)$ and the smallest positive value of b is $b=30$. The second smallest value of b is obtained if we consider the unbroken line as having been moved to the right by one full cycle plus $30°=360+30=390$.
    (b) The amplitude of 2 means $c=-2$. The solid line is the reflected sine curve shifted right by 210° so $d=210$.

13. (a) Amplitude is 3, period is $\frac{2\pi}{\pi}=2$.
    (b) See answers in Sadler.

14. (a) Amplitude is 5, period is $\frac{2\pi}{\pi/2}=4$.
    (b) See answers in Sadler.

15. See answers in Sadler. Curves are the same as $y=\tan x$ with a vertical dilation factor of 2. The second curve is the same shape as the first, but phase-shifted 45° to the left.

16. See answers in Sadler. Amplitude of curves is 3 and period is π. The second curve is the same shape as the first, but phase-shifted $\frac{\pi}{3}$ to the right.

Exercise 2C

1. $190°$ is in the 3rd quadrant where tan is **positive**.
2. $310°$ is in the 4th quadrant where cos is **positive**.
3. $-190°$ is in the 2nd quadrant wherre tan is **negative**.
4. $-170°$ is in the 3rd quadrant where sin is **negative**.
5. $555° = 360° + 195°$ so it is in the 3rd quadrant where sin is **negative**.
6. $190°$ is in the 3rd quadrant where cos is **negative**.
7. $\frac{\pi}{10}$ is in the 1st quadrant where tan is **positive**.
8. $\frac{4\pi}{5}$ is in the 2nd quadrant where sin is **positive**.
9. $\frac{\pi}{10}$ is in the 1st quadrant where cos is **positive**.
10. $-\frac{\pi}{5}$ is in the 4th quadrant where sin is **negative**.
11. $\frac{9\pi}{10}$ is in the 2nd quadrant where cos is **negative**.
12. $\frac{13\pi}{5} = 2\pi + \frac{3\pi}{5}$ so it is in the 2nd quadrant where tan is **negative**.
13. $140° = 180° - 40°$ so it makes an angle of $40°$ with the $x-$axis and is in the 2nd quadrant where sin is positive, so $\sin 140° = \sin 40°$.
14. $250° = 180° + 70°$ so it makes an angle of $70°$ with the $x-$axis and is in the 3rd quadrant where sin is negative, so $\sin 250° = -\sin 70°$.
15. $340° = 360° - 20°$ so it makes an angle of $20°$ with the $x-$axis and is in the 4th quadrant where sin is negative, so $\sin 340° = -\sin 20°$.
16. $460° = 360° + 100° = 360° + 180° - 80°$ so it makes an angle of $80°$ with the $x-$axis and is in the 2nd quadrant where sin is positive, so $\sin 460° = \sin 80°$.
17. $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$ so it makes an angle of $\frac{\pi}{6}$ with the $x-$axis and is in the 2nd quadrant where sin is positive, so $\sin\frac{5\pi}{6} = \sin\frac{\pi}{6}$.
18. $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ so it makes an angle of $\frac{\pi}{6}$ with the $x-$axis and is in the 3rd quadrant where sin is negative, so $\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6}$.
19. $\frac{11\pi}{5} = 2\pi + \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 1st quadrant where sin is positive, so $\sin\frac{11\pi}{5} = \sin\frac{\pi}{5}$.
20. $-\frac{\pi}{5}$ makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 4th quadrant where sin is negative, so $\sin -\frac{\pi}{5} = -\sin\frac{\pi}{5}$.
21. $100° = 180° - 80°$ so it makes an angle of $80°$ with the $x-$axis and is in the 2nd quadrant where cos is negative, so $\cos 100° = -\cos 80°$.
22. $200° = 180° + 20°$ so it makes an angle of $20°$ with the $x-$axis and is in the 3rd quadrant where cos is negative, so $\cos 200° = -\cos 20°$.
23. $300° = 360° - 60°$ so it makes an angle of $60°$ with the $x-$axis and is in the 4th quadrant where cos is positive, so $\cos 300° = \cos 60°$.
24. $-300° = -360° + 60°$ so it makes an angle of $60°$ with the $x-$axis and is in the 1st quadrant where cos is positive, so $\cos -300° = \cos 60°$.
25. $\frac{4\pi}{5} = \pi - \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 2nd quadrant where cos is negative, so $\cos\frac{4\pi}{5} = -\cos\frac{\pi}{5}$.
26. $\frac{9\pi}{10} = \pi - \frac{\pi}{10}$ so it makes an angle of $\frac{\pi}{10}$ with the $x-$axis and is in the 2nd quadrant where cos is negative, so $\cos\frac{9\pi}{10} = -\cos\frac{\pi}{10}$.
27. $\frac{11\pi}{10} = \pi + \frac{\pi}{10}$ so it makes an angle of $\frac{\pi}{10}$ with the $x-$axis and is in the 3rd quadrant where cos is negative, so $\cos\frac{11\pi}{10} = -\cos\frac{\pi}{10}$.
28. $\frac{21\pi}{10} = 2\pi + \frac{\pi}{10}$ so it makes an angle of $\frac{\pi}{10}$ with the $x-$axis and is in the 1st quadrant where cos is positive, so $\cos\frac{21\pi}{10} = \cos\frac{\pi}{10}$.
29. $100° = 180° - 80°$ so it makes an angle of $80°$ with the $x-$axis and is in the 2nd quadrant where tan is negative, so $\tan 100° = -\tan 80°$.
30. $200° = 180° + 20°$ so it makes an angle of $20°$ with the $x-$axis and is in the 3rd quadrant where tan is positive, so $\tan 200° = \tan 20°$.
31. $-60°$ makes an angle of $60°$ with the $x-$axis and is in the 4th quadrant where tan is negative, so $\tan -60° = -\tan 60°$.
32. $-160° = -180° + 20°$ so it makes an angle of $20°$ with the $x-$axis and is in the 2nd quadrant where tan is positive, so $\tan -160° = \tan 20°$.
33. $\frac{6\pi}{5} = \pi + \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 3rd quadrant where tan is positive, so $\tan\frac{6\pi}{5} = \tan\frac{\pi}{5}$.
34. $-\frac{6\pi}{5} = -\pi - \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 2nd quadrant where tan is negative, so $\tan -\frac{6\pi}{5} = -\tan\frac{\pi}{5}$.
35. $\frac{11\pi}{5} = 2\pi + \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 1st quadrant where tan is positive, so $\tan\frac{11\pi}{5} = \tan\frac{\pi}{5}$.
36. $-\frac{21\pi}{5} = -4\pi - \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 4th quadrant where tan is negative, so $\tan -\frac{21\pi}{5} = -\tan\frac{\pi}{5}$.
37. $300° = 360° - 60°$ and is in the 4th quadrant so
$$\begin{aligned}\sin 300° &= -\sin 60° \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

38. $210° = 180° + 30°$ and is in the 3rd quadrant so

$$\tan 210° = \tan 30° = \frac{1}{\sqrt{3}} \quad \left(\text{or } \frac{\sqrt{3}}{3}\right)$$

39. $240° = 180° + 60°$ and is in the 3rd quadrant so

$$\cos 240° = -\cos 60° = -\frac{1}{2}$$

40. $270° = 180° + 90°$ so $270°$ lies on the negative $y-$axis and $\cos 270° = 0$.

41. $180°$ lies on the negative $x-$axis so $\sin 180° = 0$.

42. $390° = 360° + 30°$ and is in the first quadrant so

$$\cos 390° = \cos 30° = \frac{\sqrt{3}}{2}$$

43. $-135° = -180° + 45°$ and is in the 3rd quadrant so

$$\sin -135° = -\sin 45° = -\frac{1}{\sqrt{2}} \quad \left(\text{or } -\frac{\sqrt{2}}{2}\right)$$

44. $-135° = -180° + 45°$ and is in the 3rd quadrant so

$$\cos -135° = -\cos 45° = -\frac{1}{\sqrt{2}} \quad \left(\text{or } -\frac{\sqrt{2}}{2}\right)$$

45. $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ and is in the 3rd quadrant so

$$\sin \frac{7\pi}{6} = -\sin \frac{\pi}{6} = -\frac{1}{2}$$

46. $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ and is in the 3rd quadrant so

$$\cos \frac{7\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

47. $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ and is in the 3rd quadrant so

$$\tan \frac{7\pi}{6} = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \quad \left(\text{or } \frac{\sqrt{3}}{3}\right)$$

48. $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ and is in the 4th quadrant so

$$\sin \frac{7\pi}{4} = -\sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}} \quad \left(\text{or } -\frac{\sqrt{2}}{2}\right)$$

49. $-\frac{7\pi}{4} = -2\pi + \frac{\pi}{4}$ and is in the 1st quadrant so

$$\cos -\frac{7\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \left(\text{or } \frac{\sqrt{2}}{2}\right)$$

50. 6π lies on the positive $x-$axis so $\tan 6\pi = \tan 0 = 0$.

51. $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$ so it lies on the positive $y-$axis and $\sin \frac{5\pi}{2} = \sin \frac{pi}{2} = 1$

52. $-\frac{7\pi}{3} = -2\pi - \frac{\pi}{3}$ and is in the 4th quadrant so

$$\cos -\frac{7\pi}{3} = \cos \frac{\pi}{3} = \frac{1}{2}$$

Exercise 2D

1. There will be a solution in the 1st and 4th quadrants (where cos is positive). $\cos 60° = \frac{1}{2}$ so $x = 60°$ or $x = 360 - 60 = 300°$.

2. There will be a solution in the 3rd and 4th quadrants (where sin is negative). $\sin 30° = \frac{1}{2}$ so $x = 180 + 30 = 210°$ or $x = 360 - 30 = 330°$.

3. There will be a solution in the 1st and 3rd quadrants (where tan is positive). $\tan 45° = 1$ so $x = 45°$ or $x = 180 + 45 = 225°$.

4. There will be a solution in the 3rd and 4th quadrants (where sin is negative). $\sin 45° = \frac{1}{\sqrt{2}}$ so $x = 180 + 45 = 225°$ or $x = 360 - 45 = 315°$.

5. There will be a solution in the 1st and 2nd quadrants (where sin is positive). $\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ so $x=\frac{\pi}{4}$ or $x=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$.

6. There will be a solution in the 2nd and 3rd quadrants (where cos is negative). $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ so $x=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$ or $x=\pi+\frac{\pi}{4}=\frac{5\pi}{4}$.

7. There will be a solution in the 2nd and 4th quadrants (where tan is negative). $\tan\frac{\pi}{4}=1$ so $x=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$ or $x=2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$.

8. There will be a solution in the 1st and 3rd quadrants (where tan is positive). $\tan\frac{\pi}{3}=\sqrt{3}$ so $x=\frac{\pi}{3}$ or $x=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$.

9. There will be a solution in the 1st and 4th quadrants (where cos is positive). $\cos 30°=\frac{\sqrt{3}}{2}$ so $x=30°$ or $x=-30°$.

10. There will be a solution in the 3rd and 4th quadrants (where sin is negative). $\sin 90°=1$ so $x=-180+90=-90°$ or $x=-90°$ (i.e. the same single solution).

11. There will be a solution in the 2nd and 4th quadrants (where tan is negative). $\tan 30°=\frac{1}{\sqrt{3}}$ so $x=180-30=150°$ or $x=-30°$.

12. sin is zero for angles that fall on the $x-$axis, so $x=-180$ or $x=0$ or $x=180$.

13. There will be a solution in the 1st and 2nd quadrants (where sin is positive). $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ so $x=\frac{\pi}{3}$ or $x=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$.

14. There will be a solution in the 2nd and 3rd quadrants (where cos is negative). $\cos\frac{\pi}{3}=\frac{1}{2}$ so $x=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$ or $x=-\pi+\frac{\pi}{3}=-\frac{2\pi}{3}$.

15. There will be a solution in the 1st and 2nd quadrants (where sin is positive). $\sin\frac{\pi}{6}=\frac{1}{2}$ so $x=\frac{\pi}{6}$ or $x=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$.

16. cos is zero for angles that fall on the $y-$axis, so $x=\frac{\pi}{2}$ or $x=-\frac{\pi}{2}$.

17. If $0\le x\le 180°$ then $0\le 2x\le 360°$. $2x$ must lie in the 1st or 3rd quadrant (where tan is positive). $\tan 30°=\frac{1}{\sqrt{3}}$ so

$$\begin{aligned} 2x &= 30 &\quad\text{or}\quad&& 2x &= 180+30=210\\ x &= 15° &&& x &= 105° \end{aligned}$$

18. If $0\le x\le\pi$ then $0\le 4x\le 4\pi$. $4x$ must lie in the 1st or 4th quadrant (where cos is positive). $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$ so:

$$\begin{aligned} 4x &= \frac{\pi}{6} &\quad\text{or}\quad&& 4x &= 2\pi-\frac{\pi}{6}\\ x &= \frac{\pi}{24} &&& &= \frac{11\pi}{6}\\ &&&& x &= \frac{11\pi}{24} \end{aligned}$$

$$\begin{aligned} \text{or}\quad 4x &= 2\pi+\frac{\pi}{6} &\quad\text{or}\quad&& 4x &= 4\pi-\frac{\pi}{6}\\ &= \frac{13\pi}{6} &&& &= \frac{23\pi}{6}\\ x &= \frac{13\pi}{24} &&& x &= \frac{23\pi}{24} \end{aligned}$$

19. If $-90°\le x\le 90°$ then $-270°\le 3x\le 270°$. $3x$ must lie in the 1st or 2nd quadrant (where sin is positive). $\sin 30°=\frac{1}{2}$ so:

$$\begin{aligned} 3x &= -180-30 &\text{ or }&& 3x &= 30° &\text{ or }&& 3x &= 180-30\\ &= -210° &&& x &= 10° &&& &= 150°\\ x &= -70° &&& &&&& x &= 50° \end{aligned}$$

20. First rearrange the equation:

$$\begin{aligned} 2\sqrt{3}\sin 2x &= 3\\ \sin 2x &= \frac{3}{2\sqrt{3}}\\ \sin 2x &= \frac{\sqrt{3}}{2} \end{aligned}$$

If $0\le x\le 2\pi$ then $0\le 2x\le 4\pi$. $2x$ must lie in the 1st or 2nd quadrant (where sin is positive). $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ so:

$$\begin{aligned} 2x &= \frac{\pi}{3} &\quad\text{or}\quad&& 2x &= \pi-\frac{\pi}{3}\\ x &= \frac{\pi}{6} &&& &= \frac{2\pi}{3}\\ &&&& x &= \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} \text{or}\quad 2x &= 2\pi+\frac{\pi}{3} &\quad\text{or}\quad&& 2x &= 3\pi-\frac{\pi}{3}\\ &= \frac{7\pi}{3} &&& &= \frac{8\pi}{3}\\ x &= \frac{7\pi}{6} &&& x &= \frac{4\pi}{3} \end{aligned}$$

21. First rearrange the equation:

$$\begin{aligned} 2\cos 3x+\sqrt{3} &= 0\\ 2\cos 3x &= -\sqrt{3}\\ \cos 3x &= -\frac{\sqrt{3}}{2} \end{aligned}$$

If $0\le x\le 2\pi$ then $0\le 3x\le 6\pi$. $3x$ must lie in the 2nd or 3rd quadrant (where cos is negative). $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$ so:

$$\begin{aligned} 3x &= \pi-\frac{\pi}{6} &\quad\text{or}\quad&& 3x &= \pi+\frac{\pi}{6}\\ &= \frac{5\pi}{6} &&& &= \frac{7\pi}{6}\\ x &= \frac{5\pi}{18} &&& x &= \frac{7\pi}{18} \end{aligned}$$

$$\begin{aligned} \text{or}\quad 3x &= 3\pi-\frac{\pi}{6} &\quad\text{or}\quad&& 3x &= 3\pi+\frac{\pi}{6}\\ &= \frac{17\pi}{6} &&& &= \frac{19\pi}{6}\\ x &= \frac{17\pi}{18} &&& x &= \frac{19\pi}{18} \end{aligned}$$

$$\text{or} \quad 3x = 5\pi - \frac{\pi}{6} \qquad \text{or} \quad 3x = 5\pi + \frac{\pi}{6}$$
$$= \frac{29\pi}{6} \qquad = \frac{31\pi}{6}$$
$$x = \frac{29\pi}{18} \qquad x = \frac{31\pi}{18}$$

22. Using the null factor law:

$$\sin x + 1 = 0 \quad \text{or} \quad 2\sin x - 1 = 0$$
$$\sin x = -1 \qquad 2\sin x = 1$$
$$x = \frac{3\pi}{2} \qquad \sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}$$
$$\text{or} \quad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

23. $\sin^2 x = \frac{1}{2}$

$$\sin x = \pm\frac{1}{\sqrt{2}}$$

This gives solutions in all 4 quadrants. $\sin 45° = \frac{1}{\sqrt{2}}$ so:

$$x = 45°$$
$$\text{or} \quad x = 180 - 45 = 135°$$
$$\text{or} \quad x = 180 + 45 = 225°$$
$$\text{or} \quad x = 360 - 45 = 315°$$

24. $4\cos^2 x - 3 = 0$

$$4\cos^2 x = 3$$
$$\cos^2 x = \frac{3}{4}$$
$$\cos x = \pm\frac{\sqrt{3}}{2}$$

This gives solutions in all 4 quadrants. $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ so:

$$x = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$
$$\text{or} \quad x = -\frac{\pi}{6}$$
$$\text{or} \quad x = \frac{\pi}{6}$$
$$\text{or} \quad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

25. $\sin x = 0$ or $2\cos x - 1 = 0$

$$x = 0 \qquad 2\cos x = 1$$
$$\text{or } x = 180° \qquad \cos x = \frac{1}{2}$$
$$\text{or } x = -180° \qquad x = 60°$$
$$\text{or } x = -60°$$

26. $\tan x = 1.5$ has solutions in the 1st and 3rd quadrant where tan is positive. $x = 0.98$ is in the 1st quadrant so there must be another solution at $x = \pi + 0.98 = 3.14 + 0.98 = 4.12$.

27. (a) $(2p-1)(p+1) = 2p^2 + 2p - p - 1$
$= 2p^2 + p - 1$

(b) By substituting $p = \cos x$ and comparing with the previous answer we see we can factorise this:

$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$

Now using the null factor law:

$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0$$
$$2\cos x = 1 \qquad \cos x = -1$$
$$\cos x = \frac{1}{2} \qquad x = \pi$$
$$x = \frac{\pi}{3} \qquad \text{or } x = -\pi$$
$$\text{or } x = -\frac{\pi}{3}$$

28. If $0 \le x \le 2\pi$ then $0 + \frac{\pi}{3} \le x + \frac{\pi}{3} \le 2\pi + \frac{\pi}{3}$ i.e. $\frac{\pi}{3} \le x + \frac{\pi}{3} \le \frac{7\pi}{3}$

$x + \frac{\pi}{3}$ must be in the 1st or 2nd quadrant (where sin is positive), and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ so:

$$x + \frac{\pi}{3} = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad \text{or} \quad x + \frac{\pi}{3} = 2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$$
$$x = \frac{3\pi}{4} - \frac{\pi}{3} = \frac{9\pi}{12} - \frac{4\pi}{12} = \frac{5\pi}{12} \qquad x = \frac{9\pi}{4} - \frac{\pi}{3} = \frac{27\pi}{12} - \frac{4\pi}{12} = \frac{23\pi}{12}$$

(Note: we can't use $x + \frac{\pi}{3} = \frac{\pi}{4}$ because it is outside the specified interval of possible values for x.)

Miscellaneous Exercise 2

1. $$\begin{aligned}\overrightarrow{\mathrm{AP}} &= \frac{5}{7}\overrightarrow{\mathrm{AB}}\\ \overrightarrow{\mathrm{OP}} &= \overrightarrow{\mathrm{OA}} + \frac{5}{7}(\overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}})\\ &= \frac{2}{7}\overrightarrow{\mathrm{OA}} + \frac{5}{7}\overrightarrow{\mathrm{OB}}\\ &= \frac{2}{7}(19\mathbf{i} + 18\mathbf{j}) + \frac{5}{7}(26\mathbf{i} - 17\mathbf{j})\\ &= \frac{38}{7}\mathbf{i} + \frac{36}{7}\mathbf{j} + \frac{130}{7}\mathbf{i} - \frac{85}{7}\mathbf{j}\\ &= \frac{38+130}{7}\mathbf{i} + \frac{36-85}{7}\mathbf{j}\\ &= 24\mathbf{i} - 7\mathbf{j}\\ |\overrightarrow{\mathrm{OP}}| &= \sqrt{24^2 + 7^2}\\ &= 25 \text{ units}\end{aligned}$$

2. (a) $8^3 \times 8^4 = 8^{3+4} = 8^7$
   (b) $\sqrt{8} = 8^{\frac{1}{2}}$
   (c) $64 = 8^2$
   (d) $2 = \sqrt[3]{8} = 8^{\frac{1}{3}}$
   (e) $4 = 2^2 = \left(8^{\frac{1}{3}}\right)^2 = 8^{\frac{2}{3}}$
   (f) $0.125 = \frac{1}{8} = 8^{-1}$

3. Substitute $-3 + 7\mathrm{i}$ for z:

$$\begin{aligned}\text{L.H.S.:}\quad z^2 &= (-3 + 7\mathrm{i})^2\\ &= 9 - 42\mathrm{i} + 49\mathrm{i}^2\\ &= 9 - 49 - 42\mathrm{i}\\ &= -40 - 42\mathrm{i}\\ &= \text{R.H.S.}\end{aligned}$$

□

It should be clear that if $z = -3+7\mathrm{i}$ is a solution then $z = -(-3+7\mathrm{i}) = 3 - 7\mathrm{i}$ is also a solution.

How would we go about finding these solutions without first being told one of them? Let the solution be $z = a + b\mathrm{i}$, with a and b real, then:

$$\begin{aligned}(a + b\mathrm{i})^2 &= -40 - 42\mathrm{i}\\ a^2 + 2ab\mathrm{i} + b^2\mathrm{i}^2 &= -40 - 42\mathrm{i}\\ a^2 - b^2 + 2ab\mathrm{i} &= -40 - 42\mathrm{i}\\ 2ab &= -42\\ ab &= -21\\ b &= -\frac{21}{a}\\ a^2 - b^2 &= -40\\ a^2 - \left(-\frac{21}{a}\right)^2 &= -40\\ a^2 - \frac{441}{a^2} &= -40\\ a^4 - 441 &= -40a^2\\ a^4 + 40a^2 - 441 &= 0\\ (a^2 + 49)(a^2 - 9) &= 0\\ a^2 &= 9\\ a &= \pm 3\\ b &= -\frac{21}{a}\\ &= \mp 7\end{aligned}$$

(We would not need to consider $a^2 + 49 = 0$ because this has no real solution and we stipulated a was real.)

4. (a) $8 = 2^3$ so $\log_2 8 = 3$
   (b) $25 = 5^2$ so $\log_5 25 = 2$
   (c) $0.2 = \frac{1}{5} = 5^{-1}$ so $\log_5 0.2 = -1$
   (d) $\sqrt{2} = 2^{\frac{1}{2}}$ so $\log_2 \sqrt{2} = \frac{1}{2}$
   (e) $1000 = 10^3$ so $\log 1000 = 3$
   (f) $a^3 \times a^7 = a^{10}$ so $\log_a(a^3 \times a^7) = 10$

5. Rearrange the equation first:

$$\begin{aligned}\sqrt{2}\sin 5x &= 1\\ \sin 5x &- \frac{1}{\sqrt{2}}\end{aligned}$$

If $0 \le x \le \pi$ then $0 \le 5x \le 5\pi$. $5x$ must be in the 1st or 2nd quadrant (where sin is positive), and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ so:

$$5x = \frac{\pi}{4},\quad x = \frac{\pi}{20} \qquad\text{or}\qquad 5x = \pi - \frac{\pi}{4} = \frac{3\pi}{4},\quad x = \frac{3\pi}{20}$$

$$\text{or}\quad 5x = 2\pi + \frac{\pi}{4} = \frac{9\pi}{4},\quad x = \frac{9\pi}{20} \qquad\text{or}\qquad 5x = 3\pi - \frac{\pi}{4} = \frac{11\pi}{4},\quad x = \frac{11\pi}{20}$$

$$\text{or}\quad 5x = 4\pi + \frac{\pi}{4} = \frac{17\pi}{4},\quad x = \frac{17\pi}{20} \qquad\text{or}\qquad 5x = 5\pi - \frac{\pi}{4} = \frac{19\pi}{4},\quad x = \frac{19\pi}{20}$$

6. (a) $\bar{z} = -5\sqrt{2}\mathrm{i}$
   (b) $z^2 = (5\sqrt{2}\mathrm{i})^2 = 25 \times 2 \times \mathrm{i}^2 = -50$
   (c) $(1 + z)^2 = 1 + 2z + z^2 = 1 + 10\sqrt{2}\mathrm{i} - 50 = -49 + 10\sqrt{2}\mathrm{i}$

7. (a) $$\begin{aligned}z + w &= 4 + 7\mathrm{i} + 2 - \mathrm{i}\\ &= 6 + 6\mathrm{i}\end{aligned}$$
   (b) $$\begin{aligned}zw &= (4 + 7\mathrm{i})(2 - \mathrm{i})\\ &= 8 - 4\mathrm{i} + 14\mathrm{i} - 7\mathrm{i}^2\\ &= 8 + 7 + 10\mathrm{i}\\ &= 15 + 10\mathrm{i}\end{aligned}$$
   (c) $\bar{z} = 4 - 7\mathrm{i}$

(d) $\bar{z}\bar{w} = (4-7\mathrm{i})(2+\mathrm{i})$
$= 8+4\mathrm{i}-14\mathrm{i}-7\mathrm{i}^2$
$= 8+7-10\mathrm{i}$
$= 15-10\mathrm{i}$

(e) $z^2 = (4+7\mathrm{i})^2$
$= 16+56\mathrm{i}+49\mathrm{i}^2$
$= 16+56\mathrm{i}-49$
$= -33+56\mathrm{i}$

(f) $(zw)^2 = (15+10\mathrm{i})^2$
$= 225+300\mathrm{i}+100\mathrm{i}^2$
$= 225+300\mathrm{i}-100$
$= 125+300\mathrm{i}$

(g) $p = \mathrm{Re}(\bar{z}) + \mathrm{Im}(\bar{w})\mathrm{i}$
$p = \mathrm{Re}(z) - \mathrm{Im}(w)\mathrm{i}$
$= 4+\mathrm{i}$

8. (a) $(2,3)$

(b) $(-5,6)$

(c) $(0,7)$

(d) $(3,0)$

(e) $(3,8)+(-2,1)=(1,9)$

(f) $(3,-5)+(3,5)=(6,0)$

(g) $(5,3)-(2,0)=(3,3)$

(h) $(2,7)-(2,-7)=(0,14)$

(i) $(0,2)\times(3,5) = (0\times3-2\times5, 0\times5+2\times3) = (-10,6)$

(j) $(-3,1)\times(-3,-1) = ((-3)^2+(1)^2,0) = (10,0)$

(k) $$\begin{aligned}(3,0)\div(2,-4) &= \frac{3}{2-4\mathrm{i}}\times\frac{2+4\mathrm{i}}{2+4\mathrm{i}}\\ &= \frac{6+12\mathrm{i}}{2^2+4^2}\\ &= \frac{6+12\mathrm{i}}{20}\\ &= 0.3+0.6\mathrm{i}\\ &= (0.3,0.6)\end{aligned}$$

(l) $$\begin{aligned}(3,-8)\div(3,8) &= \frac{3-8\mathrm{i}}{3+8\mathrm{i}}\times\frac{3-8\mathrm{i}}{3-8\mathrm{i}}\\ &= \frac{3^3-48\mathrm{i}+8^2\mathrm{i}^2}{3^3+8^2}\\ &= \frac{9-48\mathrm{i}-64}{9+64}\\ &= \frac{-55-48\mathrm{i}}{73}\\ &= \left(-\frac{55}{73},-\frac{48}{73}\right)\end{aligned}$$

9. If one solution is $x = 2+3\mathrm{i}$ then

$$\begin{aligned}(2+3\mathrm{i})^2+b(2+3\mathrm{i})+c&=0\\ (4+12\mathrm{i}-9)+b(2+3\mathrm{i})+c&=0\\ -5+12\mathrm{i}+2b+3b\mathrm{i}+c&=0\\ (-5+2b+c)+(12+3b)\mathrm{i}&=0\\ 12+3b&=0\\ b&=-4\\ -5+2b+c&=0\\ -5-8+c&=0\\ c&=13\\ x^2-4x+13&=0\end{aligned}$$

10. First factor the equation:

$$\begin{aligned}2\cos^2 x-\cos x-1&=0\\ (\cos x-1)(2\cos x+1)&=0\end{aligned}$$

Now use the null factor law:

$$\begin{aligned}\cos x-1&=0 \quad\text{or}\quad 2\cos x+1=0\\ \cos x&=1 \qquad\qquad 2\cos x=-1\\ x&=0 \qquad\qquad\ \cos x=-\frac{1}{2}\end{aligned}$$

$$\begin{aligned}x&=\pi-\frac{\pi}{3}\\ &=\frac{2\pi}{3}\\ \text{or}\quad x&=-\pi+\frac{\pi}{3}\\ &=-\frac{2\pi}{3}\end{aligned}$$

11. $$\begin{aligned}3\sin x^\circ+1&=0\\ 3\sin x^\circ&=-1\\ \sin x^\circ&=-\frac{1}{3}\end{aligned}$$

Solutions are in the 3rd and 4th quadrant where sin is negative. $\sin 19.5^\circ = \frac{1}{3}$

$$\begin{aligned}x&=180+19.5\\ &=199.5^\circ\\ \text{or}\quad x&=360-19.5\\ &=340.5^\circ\\ \text{or}\quad x&=540+19.5\\ &=559.5^\circ\\ \text{or}\quad x&=720-19.5\\ &=700.5^\circ\end{aligned}$$

12. The period is π.

$$\begin{aligned}\frac{2\pi}{a}&=\pi\\ 2\pi&=a\pi\\ a&=2\end{aligned}$$

The solid line is phase shifted to the left $\frac{\pi}{3}$ so $b=\frac{\pi}{3}$.