Chapter 5

Exercise 5A

1. $\frac{dy}{dx} = 2(4x^{2-1}) = 8x$

2. $\frac{dy}{dx} = 2(3x^{2-1}) + 1(7x^{1-1}) = 6x + 7$

3. $\frac{dy}{dx} = 1(12x^0) - 3(5x^2) = 12 - 15x^2$

4. $\frac{d}{dx}(6x^3) = 3(6x^{3-1}) = 18x^2$

5. $\frac{d}{dx}(6x^3 + 3) = 3(6x^{3-1}) + 0 = 18x^2$

6. $\frac{d}{dx}(3x^3 - x + 1) = 3(3x^{3-1}) - 1(x^{1-1}) + 0 = 9x^2 - 1$

7. $f'(x) = 0$

8. $f'(x) = 2x - 3(4x^2) + 1 = 2x - 12x^2 + 1$

9. $f'(x) = 0 + 1 + 2x + 3x^2 = 1 + 2x + 3x^2$

10. $$\begin{aligned}\frac{dy}{dx} &= (x-2)\frac{d}{dx}(x+5) + (x+5)\frac{d}{dx}(x-2)\\ &= (x-2) + (x+5)\\ &= 2x + 3\end{aligned}$$

11. $$\begin{aligned}\frac{dy}{dx} &= (2x+3)\frac{d}{dx}(3x+1) + (3x+1)\frac{d}{dx}(2x+3)\\ &= 3(2x+3) + 2(3x+1)\\ &= 6x + 9 + 6x + 2\\ &= 12x + 11\end{aligned}$$

12. $$\begin{aligned}\frac{dy}{dx} &= (x^2-5)\frac{d}{dx}(x+7) + (x+7)\frac{d}{dx}(x^2-5)\\ &= (x^2-5) + 2x(x+7)\\ &= x^2 - 5 + 2x^2 + 14x\\ &= 3x^2 + 14x - 5\end{aligned}$$

13. $$\begin{aligned}\frac{dy}{dx} &= 6x\\ \text{at } x = 2, \quad \frac{dy}{dx} &= 6 \times 2\\ &= 12\end{aligned}$$

14. $$\begin{aligned}\frac{dy}{dx} &= 6x^2\\ \text{at } x = -1, \quad \frac{dy}{dx} &= 6(-1)^2\\ &= 6\end{aligned}$$

15. $$\begin{aligned}\frac{dy}{dx} &= 1(x^2-1) + 2x(x-2)\\ &= x^2 - 1 + 2x^2 - 4x\\ &= 3x^2 - 4x - 1\end{aligned}$$
at $x = 3$,
$$\begin{aligned}\frac{dy}{dx} &= 3(3^2) - 4(3) - 1\\ &= 27 - 12 - 1\\ &= 14\end{aligned}$$

16. This question as it stands would be simplest done using the Chain Rule (see the following section in the text). To answer it using only the product rule there are a couple of approaches that could be used. The simplest, and the one appropriate at this stage of learning, is to first simplify and expand the square factor.
$$\begin{aligned}y &= (x+3)(x-2x+1)^2\\ &= (x+3)(-x+1)^2\\ &= (x+3)(x^2-2x+1)\\ \frac{dy}{dx} &= 1(x^2-2x+1) + (2x-2)(x+3)\\ &= x^2 - 2x + 1 + 2x^2 + 6x - 2x - 6\\ &= 3x^2 + 2x - 5\end{aligned}$$
at $x = 2$,
$$\begin{aligned}\frac{dy}{dx} &= 3(2)^2 + 2(2) - 5\\ &= 12 + 4 - 5\\ &= 11\end{aligned}$$

17. $$\begin{aligned}\frac{dy}{dx} &= 4x\\ 4x &= -8\\ x &= -2\\ y &= 2x^2\\ &= 2(-2)^2\\ &= 8\end{aligned}$$
The curve has a gradient of -8 at $(-2, 8)$.

18. $$\begin{aligned}\frac{dy}{dx} &= 3x^2 - 7\\ 3x^2 - 7 &= 5\\ 3x^2 &= 12\\ x^2 &= 4\\ x &= \pm 2\end{aligned}$$
At $x = 2$
$$\begin{aligned}y &= (2)^3 - 7(2)\\ &= 8 - 14\\ &= -6\end{aligned}$$
At $x = -2$
$$\begin{aligned}y &= (-2)^3 - 7(-2)\\ &= -8 + 14\\ &= 6\end{aligned}$$
The curve has a gradient of 5 at $(2, -6)$ and $(-2, 6)$

19. $\frac{dy}{dx} = 2x$. At $x = -2$, $\frac{dy}{dx} = 2 \times -2 = -4$. The equation of the tangent line (using the gradient-point form for the equation of a line):

$$\begin{aligned}(y - y_1) &= m(x - x_1)\\ y - 4 &= -4(x - -2)\\ y &= -4(x + 2) + 4\\ &= -4x - 8 + 4\\ &= -4x - 4\end{aligned}$$

20. $\frac{dy}{dx} = 5 - 3x^2$. At $x = 1$, $\frac{dy}{dx} = 5 - 3(1)^2 = 2$. The equation of the tangent line is:

$$\begin{aligned}(y - y_1) &= m(x - x_1)\\ y - 4 &= 2(x - 1)\\ y &= 2(x - 1) + 4\\ &= 2x - 2 + 4\\ &= 2x + 2\end{aligned}$$

21. (a) $f'(x) = 3 - 6x^2$
    (b) $f'(2) = 3 - 6(2)^2 = 3 - 24 = -21$

22. We expect $\frac{d}{dx}(x^5) = 5x^4$.

$$\begin{aligned}\frac{d}{dx}((x^2)(x^3)) &= 2x(x^3) + 3x^2(x^2)\\ &= 2x^4 + 3x^4\\ &= 5x^4\end{aligned}$$

23. The gradient of the line is 5. The gradient of the curve is

$$\frac{dy}{dx} = 3x^2 - 6x - 4$$

so the $x-$coordinate is the solution to $\frac{dy}{dx} = 5$:

$$\begin{aligned}3x^2 - 6x - 4 &= 5\\ 3x^2 - 6x - 9 &= 0\\ 3(x - 3)(x + 1) &= 0\\ x &= 3\\ \text{or} \quad x &= -1\end{aligned}$$

For $x = 3$,

$$\begin{aligned}y &= x^3 - 3x^2 - 4x + 1\\ &= 27 - 27 - 12 + 1\\ &= -11\end{aligned}$$

For $x = -1$,

$$\begin{aligned}y &= x^3 - 3x^2 - 4x + 1\\ &= -1 - 3 + 4 + 1\\ &= 1\end{aligned}$$

The curve has the same gradient as the line at $(3, -11)$ and at $(-1, 1)$.

24. For $f(x) = \frac{1}{x}$:

$$\begin{aligned}f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\\ &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}\\ &= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}\\ &= \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}\\ &= \lim_{h \to 0} \frac{\frac{-h}{x(x+h)}}{h}\\ &= \lim_{h \to 0} \frac{-1}{x(x + h)}\\ &= -\frac{1}{x^2}\end{aligned}$$

This confirms that $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$.

For $f(x) = \sqrt{x}$:

$$\begin{aligned}f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\\ &= \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}\\ &= \lim_{h \to 0} \left(\frac{\sqrt{x + h} - \sqrt{x}}{h} \times \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}\right)\\ &= \lim_{h \to 0} \frac{(x + h) - x}{h(\sqrt{x + h} + \sqrt{x})}\\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{x + h} + \sqrt{x})}\\ &= \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}}\\ &= \frac{1}{\sqrt{x} + \sqrt{x}}\\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

This confirms that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

Exercise 5B

1.
$$\begin{aligned} u &= 2x \\ v &= x - 1 \\ \frac{\mathrm{d}}{\mathrm{d}x}\frac{2x}{x-1} &= \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \\ &= \frac{2(x-1) - 1(2x)}{(x-1)^2} \\ &= \frac{2x - 2 - 2x}{x^2 - 2x + 1} \\ &= -\frac{2}{x^2 - 2x + 1} \end{aligned}$$

2.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{5x}{2x-3} &= \frac{5(2x-3) - 2(5x)}{(2x-3)^2} \\ &= \frac{10x - 15 - 10x}{(2x-3)^2} \\ &= -\frac{15}{(2x-3)^2} \end{aligned}$$

3.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{3x}{2x-1} &= \frac{3(2x-1) - 2(3x)}{(2x-1)^2} \\ &= \frac{6x - 3 - 6x}{(2x-1)^2} \\ &= -\frac{3}{(2x-1)^2} \end{aligned}$$

4.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{3x}{1-5x} &= \frac{3(1-5x) - -5(3x)}{(1-5x)^2} \\ &= \frac{3 - 15x + 15x}{(1-5x)^2} \\ &= \frac{3}{(1-5x)^2} \end{aligned}$$

5.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{5x+2}{2x-1} &= \frac{5(2x-1) - 2(5x+2)}{(2x-1)^2} \\ &= \frac{10x - 5 - 10x - 4)}{(2x-1)^2} \\ &= -\frac{9}{(2x-1)^2} \end{aligned}$$

6.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{x-6}{5-2x} &= \frac{1(5-2x) - -2(x-6)}{(5-2x)^2} \\ &= \frac{5 - 2x + 2x - 12}{(5-2x)^2} \\ &= -\frac{7}{(5-2x)^2} \end{aligned}$$

7.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{7-3x}{5+2x} &= \frac{-3(5+2x) - 2(7-3x)}{(5+2x)^2} \\ &= \frac{-15 - 6x - 14 + 6x}{(5+2x)^2} \\ &= -\frac{29}{(5+2x)^2} \end{aligned}$$

8.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{3x}{x^2-1} &= \frac{3(x^2-1) - 2x(3x)}{(x^2-1)^2} \\ &= \frac{3x^2 - 3 - 6x^2}{(x^2-1)^2} \\ &= \frac{-3x^2 - 3}{(x^2-1)^2} \\ &= -\frac{3x^2 + 3}{(x^2-1)^2} \end{aligned}$$

9.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{3x-4}{3x^2+1} &= \frac{3(3x^2+1) - 6x(3x-4)}{(3x^2+1)^2} \\ &= \frac{9x^2 + 3 - 18x^2 + 24x}{(3x^2+1)^2} \\ &= \frac{-9x^2 + 3 + 24x}{(3x^2+1)^2} \\ &= -\frac{3(3x^2 - 8x - 1)}{(3x^2+1)^2} \end{aligned}$$

10.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 5(6x - 2) \\ &= 30x - 10 \end{aligned}$$

11.
$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{\mathrm{d}p}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \\ &= 10s \times -2 \\ &= -20(3 - 2t) \\ &= 40t - 60 \end{aligned}$$

12.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}x} \\ &= (6u)(2)(2) \\ &= 24u \\ &= 24(2p - 1) \\ &= 48p - 24 \\ &= 48(2x + 1) - 24 \\ &= 96x + 24 \end{aligned}$$

13.
$$\begin{aligned} u &= 2x + 3 \\ y &= u^3 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (3u^2)(2) \\ &= 6(2x + 3)^2 \end{aligned}$$

14.
$$\begin{aligned} u &= 5 - 3x \\ y &= u^5 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (5u^4)(-3) \\ &= -15(5 - 3x)^4 \end{aligned}$$

15.
$$\begin{aligned} f'(x) &= 4(3x + 5)^3(3) \\ &= 12(3x + 5)^3 \end{aligned}$$

16.
$$\begin{aligned} f'(x) &= 4(x+5)^3(1) \\ &= 4(x+5)^3 \end{aligned}$$

17.
$$\begin{aligned} f'(x) &= 7(2x+3)^6(2) \\ &= 14(2x+3)^6 \end{aligned}$$

18.
$$\begin{aligned} f'(x) &= 3(5x^2+2)^2(10x) \\ &= 30x(5x^2+2)^2 \end{aligned}$$

19.
$$\begin{aligned} f'(x) &= 3(1-2x)^2(-2) \\ &= -6(1-2x)^2 \end{aligned}$$

20.
$$\begin{aligned} f'(x) &= 5+5(4x+1)^4(4) \\ &= 5+20(4x+1)^4 \end{aligned}$$

21.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (x-3)^5\frac{\mathrm{d}}{\mathrm{d}x}(x^2) + x^2\frac{\mathrm{d}}{\mathrm{d}x}(x-3)^5 \\ &= 2x(x-3)^5 + x^2(5(x-3)^4) \\ &= 2x(x-3)(x-3)^4 + 5x^2(x-3)^4 \\ &= (2x^2-6x)(x-3)^4 + 5x^2(x-3)^4 \\ &= (7x^2-6x)(x-3)^4 \end{aligned}$$

22.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (x+1)^3\frac{\mathrm{d}}{\mathrm{d}x}(3x) + 3x\frac{\mathrm{d}}{\mathrm{d}x}(x+1)^3 \\ &= 3(x+1)^3 + 3x(3(x+1)^2) \\ &= 3((x+1)(x+1)^2 + 3x(x+1)^2) \\ &= 3(4x+1)(x+1)^2 \end{aligned}$$

23.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (x^2+3)^4\frac{\mathrm{d}}{\mathrm{d}x}(2x) + 2x\frac{\mathrm{d}}{\mathrm{d}x}(x^2+3)^4 \\ &= 2(x^2+3)^4 + 2x(4(x^2+3)^3(2x)) \\ &= 2(x^2+3)^4 + 16x^2(x^2+3)^3 \\ &= 2(x^2+3)(x^2+3)^3 + 16x^2(x^2+3)^3 \\ &= (2x^2+6+16x^2)(x^2+3)^3 \\ &= 6(3x^2+1)(x^2+3)^3 \end{aligned}$$

24.
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}((5x-1)(x+5)) \\ &= (x+5)\frac{\mathrm{d}}{\mathrm{d}x}(5x-1) + (5x-1)\frac{\mathrm{d}}{\mathrm{d}x}(x+5) \\ &= 5(x+5) + (5x-1) \\ &= 5x+25+5x-1 \\ &= 10x+24 \end{aligned}$$

25.
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\frac{2x+3}{3x+2} \\ &= \frac{(3x+2)\frac{\mathrm{d}}{\mathrm{d}x}(2x+3) - (2x+3)\frac{\mathrm{d}}{\mathrm{d}x}(3x+2)}{(3x+2)^2} \\ &= \frac{2(3x+2)-3(2x+3)}{(3x+2)^2} \\ &= \frac{6x+4-6x-9}{(3x+2)^2} \\ &= -\frac{5}{(3x+2)^2} \end{aligned}$$

26.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(3x^2-1)^4 &= 4(3x^2-1)^3\frac{\mathrm{d}}{\mathrm{d}x}(3x^2-1) \\ &= 4(3x^2-1)^3(6x) \\ &= 24x(3x^2-1)^3 \end{aligned}$$

27.
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}(2x^2-3x+1)^3 \\ &= 3(2x^2-3x+1)^2\frac{\mathrm{d}}{\mathrm{d}x}(2x^2-3x+1) \\ &= 3(2x^2-3x+1)^2(4x-3) \\ &= 3(4x-3)(2x^2-3x+1)^2 \end{aligned}$$

28. The quotient rule might seem the obvious approach to this one, but it's easier to simplify before differentiating:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3+5x}{x} &= \frac{\mathrm{d}}{\mathrm{d}x}(x^2+5) \\ &= 2x \end{aligned}$$

Does the quotient rule give the same answer?

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3+5x}{x} &= \frac{x\frac{\mathrm{d}}{\mathrm{d}x}(x^3+5x) - (x^3+5x)\frac{\mathrm{d}}{\mathrm{d}x}(x)}{x^2} \\ &= \frac{x(3x^2+5)-(x^3+5x)}{x^2} \\ &= \frac{3x^3+5x-x^3-5x}{x^2} \\ &= \frac{2x^3}{x^2} \\ &= 2x \end{aligned}$$

29.
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+4x+3}{x+1} \\ &= \frac{(x+1)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+4x+3) - (x^2+4x+3)\frac{\mathrm{d}}{\mathrm{d}x}(x+1)}{(x+1)^2} \\ &= \frac{(x+1)(2x+4)-(x^2+4x+3)}{(x+1)^2} \\ &= \frac{2x^2+4x+2x+4-x^2-4x-3}{(x+1)^2} \\ &= \frac{x^2+2x+1}{x^2+2x+1} \\ &= 1 \end{aligned}$$

(This would be simpler if you realised that $\frac{x^2+4x+3}{x+1} = \frac{(x+3)(x+1)}{x+1}$ then simplify before differentiating.)

30.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 4(5-2x)^3(-2) \\ &= -8(5-2x)^3 \end{aligned}$$

At $x=2$ this evaluates to

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= -8(5-2(2))^3 \\ &= -8(1)^3 \\ &= -8 \end{aligned}$$

31.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4(x-3)-4x}{(x-3)^2}\\ &= \frac{4x-12-4x}{(x-3)^2}\\ &= -\frac{12}{(x-3)^2}\end{aligned}$$

At $x = 5$ this evaluates to

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{12}{((5)-3)^2}\\ &= -\frac{12}{2^2}\\ &= -3\end{aligned}$$

32.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{7x^6(x^2)-2x(x^7)}{(x^2)^2}\\ &= \frac{7x^8-2x^8}{x^4}\\ &= \frac{5x^8}{x^4}\\ &= 5x^4\end{aligned}$$

(just as expected.)

33. The gradient function is

$$\begin{aligned}y' &= 3(2x-5)^2(2)\\ &= 6(2x-5)^2\end{aligned}$$

and at $x = 2$ this evaluates to

$$\begin{aligned}y' &= 6(2(2)-5)^2\\ &= 6(-1)^2\\ &= 6\end{aligned}$$

The gradient-point form for the equation of a line:

$$\begin{aligned}(y-y_1) &= m(x-x_1)\\ y-(-1) &= 6(x-2)\\ y+1 &= 6x-12\\ y &= 6x-13\end{aligned}$$

34. Where the curve and line intersect,

$$\begin{aligned}\frac{5x^2}{x-1} &= 5x+3\\ 5x^2 &= (x-1)(5x+3)\\ &= 5x^2+3x-5x-3\\ 0 &= -2x-3\\ 2x &= -3\\ x &= -1.5\\ y &= 5x+3\\ &= 5(-1.5)+3\\ &= -4.5\end{aligned}$$

The line and curve intersect at $(-1.5, -4.5)$.

The gradient function of the curve is

$$y' = \frac{(x-1)(10x)-(5x^2)(1)}{(x-1)^2)}$$

At $x = -1.5$ this evaluates to

$$\begin{aligned}y' &= \frac{(-1.5-1)(10)(-1.5)-(5(-1.5)^2)(1)}{(-1.5-1)^2}\\ &= \frac{37.5-11.25}{(-2.5)^2}\\ &= \frac{26.25}{6.25}\\ &= \frac{105}{25}\\ &= 4.2\end{aligned}$$

35. The gradient function is

$$\begin{aligned}y' &= \frac{(x-2)(2x)-(x^2)(1)}{(x-2)^2}\\ &= \frac{2x^2-4x-x^2}{(x-2)^2}\\ &= \frac{x^2-4x}{(x-2)^2}\end{aligned}$$

At $x = 3$ this evaluates to

$$\begin{aligned}y' &= \frac{3^2-4\times 3}{(3-2)^2}\\ &= -3\end{aligned}$$

The gradient m of the normal is given by

$$\begin{aligned}-3m &= -1\\ m &= \frac{1}{3}\end{aligned}$$

Then using the gradient-point form to find the equation of the normal

$$\begin{aligned}(y-y_1) &= m(x-x_1)\\ y-9 &= \frac{1}{3}(x-3)\\ y &= \frac{x}{3}+8\end{aligned}$$

36. The gradient function is

$$\begin{aligned}y' &= (2x-5)^3(2)+(2x-1)(3(2x-5)^2(2))\\ &= 2(2x-5)^3+6(2x-1)(2x-5)^2\end{aligned}$$

Factorising:

$$\begin{aligned}y' &= 2(2x-5)^2((2x-5)+3(2x-1))\\ &= 2(2x-5)^2(2x-5+6x-3)\\ &= 2(2x-5)^2(8x-8)\\ &= 16(2x-5)^2(x-1)\end{aligned}$$

Thus the gradient is zero where

$$\begin{aligned}2x-5&=0\\x&=2.5\\\text{or } x-1&=0\\x&=1\end{aligned}$$

Substituting these values back into the original equation to find their corresponding y values:

$$\begin{aligned}y&=(2(2.5)-1)(2(2.5)-5)^3\\&=4\times 0^3\\&=0\\\text{or } y&=(2(1)-1)(2(1)-5)^3\\&=1\times(-3)^3\\&=-27\end{aligned}$$

The gradient of the curve is zero at $(2.5, 0)$ and at $(1, -27)$.

37. The gradient function is

$$\begin{aligned}y'&=\frac{(2x+1)(2x+2)-(x^2+2x+3)(2)}{(2x+1)^2}\\&=\frac{(4x^2+4x+2x+2)-(2x^2+4x+6)}{(2x+1)^2}\\&=\frac{2x^2+2x-4}{(2x+1)^2}\\&=\frac{2(x^2+x-2)}{(2x+1)^2}\\&=\frac{2(x+2)(x-1)}{(2x+1)^2}\end{aligned}$$

Thus the gradient is zero where

$$\begin{aligned}x+2&=0\\x&=-2\\\text{or } x-1&=0\\x&=1\end{aligned}$$

Substituting these values back into the original equation to find their corresponding y values:

$$\begin{aligned}y&=\frac{(-2)^2+2(-2)+3}{2(-2)+1}\\&=-1\\\text{or } y&=\frac{(1)^2+2(1)+3}{2(1)+1}\\&=2\end{aligned}$$

The gradient of the curve is zero at $(-2, -1)$ and at $(1, 2)$.

38. (a) First, find a by substituting $x=-3$ into the equation of the curve:

$$\begin{aligned}a&=\frac{5(-3)-7}{2(-3)+10}\\&=\frac{-22}{4}\\&=-5.5\end{aligned}$$

Now b is the gradient of the tangent line at $(-3, -5.5)$ and hence the gradient of the curve at that point, so we can find b by substituting $x=-3$ into the gradient function.

$$\begin{aligned}y'&=\frac{(2x+10)(5)-(5x-7)(2)}{(2x+10)^2}\\&=\frac{10x+50-10x+14}{(2x+10)^2}\\&=\frac{64}{(2x+10)^2}\\b&=\frac{64}{(2(-3)+10)^2}\\&=\frac{64}{4^2}\\&=4\end{aligned}$$

The equation of the tangent line is

$$\begin{aligned}y-y_1&=m(x-x_1)\\y-a&=b(x--3)\\y--5.5&=4(x+3)\\y+5.5&=4x+12\\y&=4x+6.5\end{aligned}$$

hence $c=6.5$.

(b) Solve $y'=b=4$ (already knowing that $x=-3$ is one solution):

$$\begin{aligned}y'&=4\\\frac{64}{(2x+10)^2}&=4\\\frac{16}{(2x+10)^2}&=1\\16&=(2x+10)^2\\2x+10&=\pm 4\\2x&=-10\pm 4\\x&=-5\pm 2\\x&=-7\\\text{or}\quad x&=-3\end{aligned}$$

Substituting $x=-7$ into the original equation:

$$\begin{aligned}y&=\frac{5(-7)-7}{2(-7)+10}\\&=\frac{-42}{-4}\\&=10.5\end{aligned}$$

The other point where the tangent to the curve is parallel to $y=bx+3$ has coordinates $(-7, 10.5)$.

39. (a) First, find a by substituting $x = 3$ and $y' = 2$ into the gradient equation and solving. (Remember, a is a constant, so its derivative is zero.)

$$\begin{aligned} y' &= \frac{(2x-11)(-6)-(a-6x)(2)}{(2x-11)^2} \\ &= \frac{-12x+66-2a+12x}{(2x-11)^2} \\ &= \frac{66-2a}{(2x-11)^2} \\ 2 &= \frac{66-2a}{(2(3)-11)^2} \\ 2 &= \frac{66-2a}{(-5)^2} \\ 2 &= \frac{66-2a}{25} \\ 66-2a &= 50 \\ -2a &= -16 \\ a &= 8 \end{aligned}$$

Now substitute this and $x = 3$ into the original equation to find b:

$$\begin{aligned} b &= \frac{8-6(3)}{2(3)-11} \\ &= \frac{-10}{-5} \\ &= 2 \end{aligned}$$

(b) Solve $y' = 2$ (already knowing that $x = 3$ is one solution):

$$\begin{aligned} y' &= 2 \\ \frac{66-2a}{(2x-11)^2} &= 2 \end{aligned}$$

Substituting $a = 8$:

$$\begin{aligned} \frac{50}{(2x-11)^2} &= 2 \\ \frac{25}{(2x-11)^2} &= 1 \\ 25 &= (2x-11)^2 \\ 2x-11 &= \pm 5 \\ 2x &= 11 \pm 5 \\ 2x &= 16 \\ x &= 8 \\ \text{or} \quad 2x &= 6 \\ x &= 3 \end{aligned}$$

Substituting $x = 8$ and $a = 8$ into the original equation:

$$\begin{aligned} y &= \frac{8-6(8)}{2(8)-11} \\ &= \frac{-40}{5} \\ &= -8 \end{aligned}$$

The other point where the curve has a gradient of 2 is at $(8, -8)$.

40. From the first curve:

$$\begin{aligned} y' &= (2x-3)^3(1) + (x+1)(3(2x-3)^2(2)) \\ &= (2x-3)^3 + 6(x+1)(2x-3)^2 \\ &= (2x-3)^2(2x-3+6x+6) \\ &= (2x-3)^2(8x+3) \end{aligned}$$

At $x = 2$:

$$\begin{aligned} c &= (2(2)-3)^2(8(2)+3) \\ &= 19 \end{aligned}$$

Thus the gradient of all three curves at $x = 2$ is 19.

From the second curve:

$$\begin{aligned} y' &= 6x - (-a(x-1)^{-2}) \\ &= 6x + \frac{a}{(x-1)^2} \end{aligned}$$

At $x = 2$:

$$\begin{aligned} 19 &= 6(2) + \frac{a}{(2-1)^2} \\ &= 12 + a \\ a &= 7 \end{aligned}$$

From the third curve:

$$\begin{aligned} y' &= \frac{(4-x)(b)-(bx+12)(-1)}{(4-x)^2} \\ &= \frac{4b-bx+bx+12}{(4-x)^2} \\ &= \frac{4b+12}{(4-x)^2} \end{aligned}$$

At $x = 2$:

$$\begin{aligned} 19 &= \frac{4b+12}{(4-2)^2} \\ &= b+3 \\ b &= 16 \end{aligned}$$

Miscellaneous Exercise 5

1. (a) Use the null factor law to give $x = 3$ or $x = -7$.

   (b)
$$\begin{aligned} 2x-5&=0 &\quad\text{or}\quad 4x+1&=0\\ 2x&=5 & 4x&=-1\\ x&=2.5 & x&=-0.25 \end{aligned}$$

   (c) First factorise then use the null factor law:
$$\begin{aligned} (x-4)(x+3)&=0\\ x&=4\\ \text{or}\quad x&=-3 \end{aligned}$$

   (d)
$$\begin{aligned} (x+7)(x-2)&=0\\ x&=-7\\ \text{or}\quad x&=2 \end{aligned}$$

   (e)
$$\begin{aligned} 5(x^2+x-12)&=0\\ 5(x+4)(x-3)&=0\\ x&=-4\\ \text{or}\quad x&=3 \end{aligned}$$

   (f)
$$\begin{aligned} 4(x^2+9x-10)&=0\\ 4(x+10)(x-1)&=0\\ x&=-10\\ \text{or}\quad x&=1 \end{aligned}$$

2. LHS:
$$\begin{aligned} \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}&=\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\times\frac{\cos\theta-\sin\theta}{\cos\theta-\sin\theta}\\ &=\frac{\cos^2\theta-2\sin\theta\cos\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta}\\ &=\frac{1-2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}\\ &=\frac{1-\sin 2\theta}{\cos 2\theta} \end{aligned}$$

□

3. From the null factor law, using the first factor:
$$\begin{aligned} 6+25\sin\theta&=0\\ \sin\theta&=-\frac{6}{25}\\ &=-0.24 \end{aligned}$$

   sine is negative in the 3rd and 4th quadrants
$$\begin{aligned} \theta&=\pi+\frac{\pi}{13}\\ &=\frac{14\pi}{13}\\ \text{or}\quad\theta&=2\pi-\frac{\pi}{13}\\ &=\frac{25\pi}{13} \end{aligned}$$

   using the second factor:
$$\begin{aligned} 1-2\cos\theta&=0\\ \cos\theta&=0.5 \end{aligned}$$

   cosine is positive in the 1st and 4th quadrants
$$\begin{aligned} \theta&=\frac{\pi}{3}\\ \text{or}\quad\theta&=\frac{5\pi}{3} \end{aligned}$$

4. (a) O is the midpoint of AB, so it has coordinates:
$$\left(\frac{1+9}{2},\frac{2+-4}{2}\right)=(5,-1)$$

   (b) The radius is the distance OA:
$$r=\sqrt{(1-5)^2+(2--1)^2}=5$$

   (c) The vector equation of a circle radius 5 centred at $(5,-1)$ is
$$\left|\mathbf{r}-\begin{pmatrix}5\\-1\end{pmatrix}\right|=5$$

5. If the equation is not to have complex solutions, b^2-4ac must be non-negative:
$$\begin{aligned} (-q)^2-4(4)(3)&\geq 0\\ q^2-48&\geq 0\\ q^2&\geq 48\\ q&\geq 4\sqrt{3}\\ \text{or}\quad q&\leq -4\sqrt{3} \end{aligned}$$

6. (a)
$$\begin{aligned} z+w&=-5+2\mathrm{i}+-3\mathrm{i}\\ &=-5-\mathrm{i} \end{aligned}$$

   (b)
$$\begin{aligned} zw&=(-5+2\mathrm{i})(-3\mathrm{i})\\ &=15\mathrm{i}+6\\ &=6+15\mathrm{i} \end{aligned}$$

   (c)
$$\bar{z}=-5-2\mathrm{i}$$

   (d)
$$\begin{aligned} \bar{z}\bar{w}&=(-5-2\mathrm{i})(3\mathrm{i})\\ &=-15\mathrm{i}+6\\ &=6-15\mathrm{i} \end{aligned}$$

   (e)
$$\begin{aligned} z^2&=(-5+2\mathrm{i})^2\\ &=25-20\mathrm{i}-4\\ &=21-20\mathrm{i} \end{aligned}$$

   (f)
$$\begin{aligned} (zw)^2&=(6+15\mathrm{i})^2\\ &=36+180\mathrm{i}-225\\ &=-189+180\mathrm{i} \end{aligned}$$

   (g)
$$\begin{aligned} p&=\mathrm{Re}(\bar{z})+\mathrm{Im}(\bar{w})\mathrm{i}\\ &=\mathrm{Re}(z)-\mathrm{Im}(w)\mathrm{i}\\ &=-5+3\mathrm{i} \end{aligned}$$

7. Let $z = a + b\text{i}$

$$\begin{aligned} 5z - \bar{z} &= -8 + 12\text{i} \\ 5(a + b\text{i}) - (a - b\text{i}) &= -8 + 12\text{i} \\ 4a + 6b\text{i} &= -8 + 12\text{i} \\ a &= -2 \\ b &= 2 \\ z &= -2 + 2\text{i} \end{aligned}$$

8.
$$\begin{aligned} (x + \text{i}y)^2 &= 96 - 40\text{i} \\ x^2 + 2xy\text{i} - y^2 &= 96 - 40\text{i} \end{aligned}$$

From the imaginary components:

$$\begin{aligned} 2xy &= -40 \\ y &= -\frac{20}{x} \end{aligned}$$

From the real components:

$$\begin{aligned} x^2 - y^2 &= 96 \\ x^2 - (-\frac{20}{x})^2 &= 96 \\ x^2 - \frac{400}{x^2} &= 96 \\ x^4 - 400 &= 96x^2 \\ x^4 - 96x^2 - 400 &= 0 \\ (x^2 - 100)(x^2 + 4) &= 0 \end{aligned}$$

The second factor has no real solutions, so we can disregard it and focus on the first.

$$\begin{aligned} x^2 - 100 &= 0 \\ x^2 &= 100 \\ x &= \pm 10 \\ y &= -\frac{20}{\pm 10} \\ &= \mp 2 \end{aligned}$$

(x, y) is $(10, -2)$ or $(-10, 2)$

9. (a)
$$\begin{aligned} (2y - 1)(y + 1) &= 2y^2 + 2y - y - 1 \\ &= 2y^2 + y - 1 \end{aligned}$$

(b)
$$\begin{aligned} 1 + \sin x &= 2\cos^2 x \\ &= 2(1 - \sin^2 x) \\ &= 2 - 2\sin^2 x \\ 2\sin^2 x + \sin x - 1 &= 0 \\ (2\sin x - 1)(\sin x + 1) &= 0 \end{aligned}$$

From the first factor:

$$\begin{aligned} 2\sin x - 1 &= 0 \\ \sin x &= \frac{1}{2} \\ x &= \frac{\pi}{6}; \frac{5\pi}{6}; -\frac{7\pi}{6}; \text{ or } -\frac{11\pi}{6} \end{aligned}$$

From the second factor:

$$\begin{aligned} \sin x + 1 &= 0 \\ \sin x &= -1 \\ x &= \frac{3\pi}{2}; \text{ or } -\frac{\pi}{2} \end{aligned}$$

10. The displacement vector from ship to yacht is

$$\begin{aligned} {}_{\text{YACHT}}\mathbf{r}_{\text{SHIP}} &= \mathbf{r}_{\text{YACHT}} - \mathbf{r}_{\text{SHIP}} \\ &= (9\mathbf{i} + 8\mathbf{j}) - (10\mathbf{i} + 5\mathbf{j}) \\ &= (-\mathbf{i} + 3\mathbf{j})\text{km} \end{aligned}$$

The velocity of the ship relative to the yacht is

$$\begin{aligned} {}_{\text{SHIP}}\mathbf{v}_{\text{YACHT}} &= \mathbf{v}_{\text{SHIP}} - \mathbf{v}_{\text{YACHT}} \\ &= (8\mathbf{i} + 7\mathbf{j}) - (12\mathbf{i} - 5\mathbf{j}) \\ &= (-4\mathbf{i} + 12\mathbf{j})\text{km/h} \end{aligned}$$

Since ${}_{\text{YACHT}}\mathbf{r}_{\text{SHIP}} = 0.25\,{}_{\text{SHIP}}\mathbf{v}_{\text{YACHT}}$, the ships will collide in a quarter of an hour, i.e. at 9:15am.

The position of the collision is

$$\begin{aligned} \mathbf{r} &= (10\mathbf{i} + 5\mathbf{j}) + 0.25(8\mathbf{i} + 7\mathbf{j}) \\ &= (12\mathbf{i} + 6.75\mathbf{j})\text{km} \end{aligned}$$

11. (a) The conjugate of w has the same real component and the opposite imaginary component: it's a reflection in the $x-$axis. Diagram B.

(b) If $z + w$ is real, then they must have opposite imaginary components. This is true for diagrams B and D.

(c) If zw is real then $\text{Re}(z) \times \text{Im}(w) + \text{Im}(z) \times \text{Re}(w) = 0$ (since the other terms that arise from the multiplication are real).

$$\begin{aligned} \text{Re}(z) \times \text{Im}(w) + \text{Im}(z) \times \text{Re}(w) &= 0 \\ \text{Re}(z) \times \text{Im}(w) &= -\text{Im}(z) \times \text{Re}(w) \\ \frac{\text{Im}(w)}{\text{Re}(w)} &= -\frac{\text{Im}(z)}{\text{Re}(z)} \end{aligned}$$

On the Argand diagram this represents points having the opposite gradient. This is true for diagrams A, B and F.

(d) Numbers with an imaginary part of 1 are shown in diagrams A and C.

(e) Numbers having the absolute value of their imaginary part equal to 1 are shown in diagrams A, B, C and D.

(f) Since w has a positive imaginary component, this is no different from part (d) above: diagrams A and C.

(g) This results in an imaginary part equal to $\text{Re}(w)$ and real part equal to $-\text{Im}(w)$, i.e. a 90° rotation. This is shown in diagram E.

(h) If we multiply $\frac{\bar{w}}{z}$ by $\frac{\bar{z}}{\bar{z}}$ the denominator will always be real, so $\frac{\bar{w}}{z}$ is real if $\bar{w}\bar{z}$ is real. This is similar to part (c) above with a similar result:

$$\begin{aligned}\mathrm{Re}(\bar{w}) \times \mathrm{Im}(\bar{z}) + \mathrm{Im}(\bar{w}) \times \mathrm{Re}(\bar{z}) &= 0\\ \mathrm{Re}(w) \times -\mathrm{Im}(z) + -\mathrm{Im}(w) \times \mathrm{Re}(z) &= 0\\ -\mathrm{Re}(w) \times \mathrm{Im}(z) &= \mathrm{Im}(w) \times \mathrm{Re}(z)\\ \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} &= -\frac{\mathrm{Im}(w)}{\mathrm{Re}(w)}\end{aligned}$$

On the Argand diagram this represents points having the opposite gradient, just as in part (c). This is true for diagrams A, B and F.

12. (a) The radius is 5. The centre has position vector $7\mathbf{i}-\mathbf{j}$ which corresponds to Cartesian coordinates $(7, -1)$.

(b) The radius is 6. $|\mathbf{r} - 7\mathbf{i} - \mathbf{j}| = |\mathbf{r} - (7\mathbf{i} + \mathbf{j})|$ The centre has position vector $7\mathbf{i} + \mathbf{j}$ which corresponds to Cartesian coordinates $(7, 1)$.

(c) The radius is $\sqrt{18} = 3\sqrt{2}$. The centre is the origin, $(0, 0)$.

(d) The radius is $\sqrt{75} = 5\sqrt{3}$. The centre is $(1, -8)$.

(e)
$$\begin{aligned}x^2 + y^2 + 2x &= 14y + 50\\ x^2 + y^2 + 2x - 14y &= 50\\ (x+1)^2 - 1 + (y-7)^2 - 49 &= 50\\ (x+1)^2 + (y-7)^2 &= 100\end{aligned}$$

The radius is $\sqrt{100} = 10$. The centre is $(-1, 7)$.

(f)
$$\begin{aligned}x^2 + 10x + y^2 &= 151 + 14y\\ x^2 + 10x + y^2 - 14y &= 151\\ (x+5)^2 - 25 + (y-7)^2 - 49 &= 151\\ (x+5)^2 + (y-7)^2 &= 225\end{aligned}$$

The radius is $\sqrt{225} = 15$. The centre is $(-5, 7)$.

13. (a)
$$\begin{aligned}3x^3 &- 11x^2 + 25x - 25\\ &= (ax - b)(x^2 + cx + 5)\\ &= ax^3 + acx^2 + 5ax - bx^2 - bcx - 5b\\ &= ax^3 + (ac - b)x^2 + (5a - bc)x - 5b\end{aligned}$$

From the x^3 term:

$$a = 3$$

From the constant term:

$$\begin{aligned}-5b &= -25\\ b &= 5\end{aligned}$$

From the x^2 term:

$$\begin{aligned}ac - b &= -11\\ 3c - 5 &= -11\\ 3c &= -6\\ c &= -2\end{aligned}$$

(b) Use the results from (a) to factor the expression

$$\begin{aligned}3x^3 - 11x^2 + 25x - 25 &= 0\\ (3x - 5)(x^2 - 2x + 5) &= 0\end{aligned}$$

From the linear factor:

$$\begin{aligned}3x - 5 &= 0\\ x &= \frac{5}{3}\end{aligned}$$

From the quadratic factor, using the quadratic formula:

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}\\ &= \frac{2 \pm \sqrt{4 - 20}}{2}\\ &= \frac{2 \pm \sqrt{-16}}{2}\\ &= \frac{2 \pm 4\mathrm{i}}{2}\\ &= 1 \pm 2\mathrm{i}\end{aligned}$$

14. (a)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 4(2x^3 - 5)^3(6x^2)\\ &= 24x^2(2x^3 - 5)^3\end{aligned}$$

(b)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (2x+1)^3(6x) + (3x^2 + 2)(3(2x+1)^2(2))\\ &= 6x(2x+1)^3 + 6(3x^2 + 2)(2x+1)^2\\ &= 6(2x+1)^2(x(2x+1) + (3x^2 + 2))\\ &= 6(2x+1)^2(2x^2 + x + 3x^2 + 2)\\ &= 6(2x+1)^2(2x^2 + x + 3x^2 + 2)\\ &= 6(2x+1)^2(5x^2 + x + 2)\end{aligned}$$

15. Working left to right:

- The curve begins with a small negative gradient
- Gradient decreases to a minimum at the first marked point
- At the second marked point the curve is horizontal, so the gradient is zero. After this the gradient continues to increase.
- At the third point the gradient reaches its local maximum and begins decreasing.
- At the fourth point the curve is horizontal, so the gradient is zero.
- As it approaches the vertical asymptote the gradient of the curve increases.
- On the other side of the asymptote the gradient decreases to zero at the last marked point, then increases again.