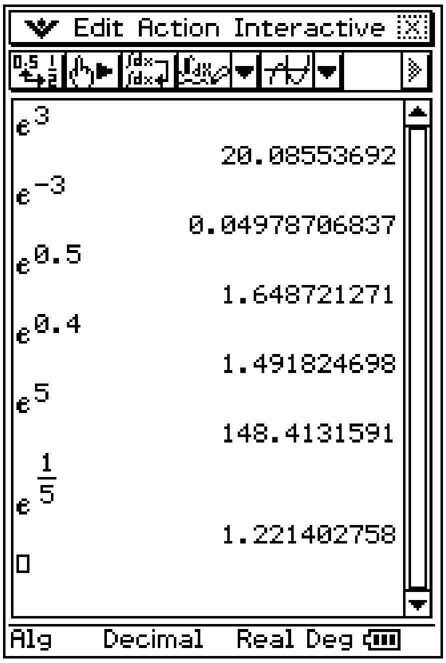
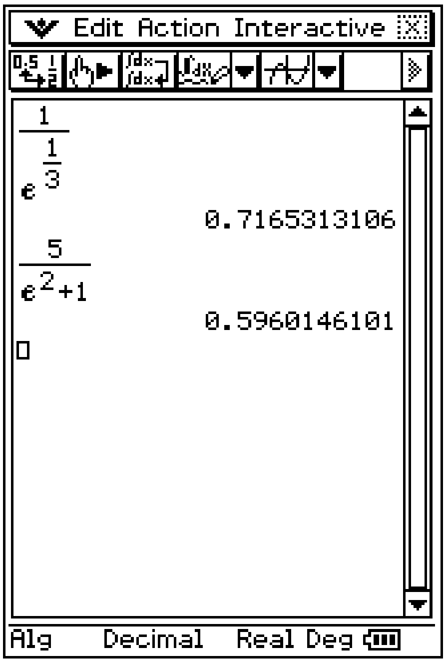
Chapter 7

Exercise 7A



1–6.



7–8.

9. (a) $t = 5 \times 60 = 300$

$$T = 10 + 65\mathrm{e}^{-0.004\times 300} = 29.6°\mathrm{C}$$

(b) $t = 5 \times 60 = 600$

$$T = 10 + 65\mathrm{e}^{-0.004\times 600} = 15.9°\mathrm{C}$$

10. $\log_\mathrm{e} \mathrm{e} = \log_\mathrm{e} \mathrm{e}^1 = 1$

11. $\log_\mathrm{e} \frac{1}{\mathrm{e}} = \log_\mathrm{e} \mathrm{e}^{-1} = -1$

12. $\log_\mathrm{e}(\mathrm{e}^3) = 3$

13. $\log_\mathrm{e} \sqrt{\mathrm{e}} = \log_\mathrm{e} \mathrm{e}^{\frac{1}{2}} = 0.5$

14. $$\begin{aligned} \mathrm{e}^{x+1} &= 7 \\ x + 1 &= \ln(7) \\ x &= \ln(7) - 1 \end{aligned}$$

15. $$\begin{aligned} \mathrm{e}^{x+3} &= 50 \\ x + 3 &= \ln(50) \\ x &= \ln(50) - 3 \end{aligned}$$

16. $$\begin{aligned} \mathrm{e}^{x-3} &= 100 \\ x - 3 &= \ln(100) \\ x &= \ln(100) + 3 \end{aligned}$$

17. $$\begin{aligned} \mathrm{e}^{2x+1} &= 15 \\ 2x + 1 &= \ln(15) \\ x &= \frac{\ln(15) - 1}{2} \end{aligned}$$

18. $$\begin{aligned} 5\mathrm{e}^{3x-1} &= 3000 \\ \mathrm{e}^{3x-1} &= 600 \\ 3x - 1 &= \ln(600) \\ x &= \frac{\ln(600) + 1}{3} \end{aligned}$$

19. $$\begin{aligned} 4\mathrm{e}^{x+2} + 3\mathrm{e}^{x+2} &= 7\,000 \\ 7\mathrm{e}^{x+2} &= 7\,000 \\ \mathrm{e}^{x+2} &= 1\,000 \\ x + 2 &= \ln(1\,000) \\ x &= \ln(1\,000) - 2 \end{aligned}$$

20. $$\begin{aligned} \mathrm{e}^2 x - 30\mathrm{e}^x &= 200 \\ (\mathrm{e}^x)^2 - 30\mathrm{e}^x &= 200 \\ y^2 - 30y = 200 &= 0 \\ (y - 10)(y - 20) &= 0 \\ y = 10 &\quad \text{or } y = 20 \\ e^x = 10e^x &\quad = 20 \\ x = \ln 10 &\quad x = \ln 20 \end{aligned}$$

21. $$\begin{aligned} A &= 2000\mathrm{e}^{-t} \\ e^{-t} &= \frac{A}{2000} \\ e^t &= \frac{2000}{A} \\ t &= \ln \frac{2000}{A} \end{aligned}$$

(a) $t = \ln \frac{2000}{1500} = 0.288$

(b) $t = \ln \frac{2000}{500} = 1.386$

(c) $t = \ln \frac{2000}{50} = 3.689$

22. (a) In 2000, $t = 10$ so

$$\begin{aligned} P &= 20\,000\,000\mathrm{e}^{0.02\times 10} \\ &= 24\,428\,000 \end{aligned}$$

(b) In 2050, $t = 60$ so

$$\begin{aligned} P &= 20\,000\,000\mathrm{e}^{0.02\times 60} \\ &= 66\,402\,000 \end{aligned}$$

23. (a) $t = 0$ so

$$\begin{aligned} N &= 5\,000\mathrm{e}^{0.55\times 0} \\ &= 5\,000 \end{aligned}$$

(b) $t = 3$ so

$$\begin{aligned} N &= 5\,000\mathrm{e}^{0.55\times 3} \\ &= 26\,000 \end{aligned}$$

(c) $t = 10$ so

$$\begin{aligned} N &= 5\,000\mathrm{e}^{0.55\times 10} \\ &= 1\,233\,000 \end{aligned}$$

24. In 2010, $t = 20$ so the company's requirement is

$$P\mathrm{e}^{0.1\times 20} = 7.39P$$

The requirement has increased from 100% of P in 1990 to 739% of P in 2010, i.e. an increase of 639%.

25. (a) $$\begin{aligned} N &= \frac{100\,000}{1 + 499\mathrm{e}^{-0.8\times 0}} \\ &= 200 \end{aligned}$$

(b) $$\begin{aligned} N &= \frac{100\,000}{1 + 499\mathrm{e}^{-0.8\times 5}} \\ &= 9\,862 \end{aligned}$$

(c) $$\begin{aligned} N &= \frac{100\,000}{1 + 499\mathrm{e}^{-0.8\times 10}} \\ &= 85\,661 \end{aligned}$$

(d) $$\begin{aligned} \lim_{t\to\infty} \frac{100\,000}{1 + 499\mathrm{e}^{-0.8t}} &= \frac{100\,000}{1} \\ &= 100\,000 \end{aligned}$$

$$\left(\text{since } \lim_{t\to\infty} \mathrm{e}^{-0.08t} = \mathrm{e}^{-\infty} = \frac{1}{\mathrm{e}^{\infty}} = \frac{1}{\infty} = 0\right)$$

Exercise 7B

1. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = e^x$
2. $\frac{\mathrm{d}}{\mathrm{d}x}2\mathrm{e}^x = 2\mathrm{e}^x$
3. $\frac{\mathrm{d}}{\mathrm{d}x}10\mathrm{e}^x = 10\mathrm{e}^x$
4. $\frac{\mathrm{d}}{\mathrm{d}x}(5x^2 + \mathrm{e}^x) = 10x + \mathrm{e}^x$
5. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x + 3x^2 + x^3) = \mathrm{e}^x + 6x + 3x^2$
6. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{5x} = 5\mathrm{e}^{5x}$
7. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{4x} = 4\mathrm{e}^{4x}$
8. $\frac{\mathrm{d}}{\mathrm{d}x}3\mathrm{e}^{4x} = 12\mathrm{e}^{4x}$
9. $\frac{\mathrm{d}}{\mathrm{d}x}3\mathrm{e}^{2x} = 6\mathrm{e}^{2x}$
10. $\frac{\mathrm{d}}{\mathrm{d}x}5\mathrm{e}^{4x} = 20\mathrm{e}^{4x}$
11. $\frac{\mathrm{d}}{\mathrm{d}x}(2\mathrm{e}^{3x} + 3\mathrm{e}^{2x}) = 6\mathrm{e}^{3x} + 6\mathrm{e}^{2x} = 6\mathrm{e}^{2x}(\mathrm{e}^x + 1)$
12. $\frac{\mathrm{d}}{\mathrm{d}x}(4e^{3x} + x^4 - 2) = 12\mathrm{e}^{3x} + 4x^3$
13. $\frac{\mathrm{d}}{\mathrm{d}x}e^{2x-4} = 2e^{2x-4}$
14. $\frac{\mathrm{d}}{\mathrm{d}x}e^{3x+1} = 3e^{3x+1}$
15. $\frac{\mathrm{d}}{\mathrm{d}x}e^{x^3} = 3x^2e^{x^3}$
16. $\frac{\mathrm{d}}{\mathrm{d}x}e^{x^2+5x-1} = (2x+5)e^{x^2+5x-1}$
17. Product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x^2\mathrm{e}^x &= 2x\mathrm{e}^x + x^2\mathrm{e}^x \\ &= x\mathrm{e}^x(2 + x) \end{aligned}$$

18. Sum and product rules:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x + x\mathrm{e}^x) &= 1 + (\mathrm{e}^x + x\mathrm{e}^x) \\ &= 1 + \mathrm{e}^x(1 + x) \end{aligned}$$

19. Product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x^3\mathrm{e}^x &= 3x^2\mathrm{e}^x + x^3\mathrm{e}^x \\ &= x^2\mathrm{e}^x(3 + x) \end{aligned}$$

20. $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^3 + \mathrm{e}^x) = 3x^2 + \mathrm{e}^x$
21. Product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x\mathrm{e}^{2x} &= \mathrm{e}^{2x} + 2x\mathrm{e}^{2x} \\ &= \mathrm{e}^{2x}(1 + 2x) \end{aligned}$$

22. Product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}2x\mathrm{e}^x &= 2\mathrm{e}^x + 2x\mathrm{e}^x \\ &= 2\mathrm{e}^x(1 + x) \end{aligned}$$

23. Quotient rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^x}{x} = \frac{x\mathrm{e}^x - \mathrm{e}^x}{x^2} = \frac{\mathrm{e}^x(x-1)}{x^2}$$

24. Product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x(1+x) = \mathrm{e}^x(1+x) + \mathrm{e}^x = \mathrm{e}^x(2+x)$$

25. Product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x(1+x)^5 = \mathrm{e}^x(1+x)^5 + 5\mathrm{e}^x(1+x)^4 = \mathrm{e}^x(1+x)^4(1+x+5) = \mathrm{e}^x(1+x)^4(6+x)$$

26. Product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x(10-x)^4 = \mathrm{e}^x(10-x)^4 + (-4)\mathrm{e}^x(10-x)^3 = \mathrm{e}^x(10-x)^3(10-x-4) = \mathrm{e}^x(10-x)^3(6-x)$$

27. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\mathrm{e}^x} = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x} = -\mathrm{e}^{-x} = -\frac{1}{\mathrm{e}^x}$

28. $$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x + 2\mathrm{e}^{2x}$$

At $x = 1$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6(1) + 2\mathrm{e}^{2(1)} = 6 + 2\mathrm{e}^2$$

29. $$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\mathrm{e}^{2x} + 2x^2\mathrm{e}^{2x} = 2x\mathrm{e}^{2x}(1+x)$$

At $x = 1$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(1)\mathrm{e}^{2(1)}(1+1) = 4\mathrm{e}^2$$

30. $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{e^x}$ (see number 27 above)

$$\begin{aligned} -\frac{1}{\mathrm{e}^x} &= -\mathrm{e} \\ \frac{1}{\mathrm{e}^x} &= \mathrm{e} \\ 1 &= \mathrm{e}^{1+x} \\ 1+x &= \ln 1 \\ &= 0 \\ x &= -1 \\ y &= \frac{1}{\mathrm{e}^{-1}} \\ &= \mathrm{e} \end{aligned}$$

The coordinates are $(-1, \mathrm{e})$.

31. First differentiate using the quotient rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x\mathrm{e}^x - 2\mathrm{e}^x}{(2x)^2} = \frac{2\mathrm{e}^x(x-1)}{4x^2} = \frac{\mathrm{e}^x(x-1)}{2x^2}$$

Now find where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\begin{aligned} \frac{\mathrm{e}^x(x-1)}{2x^2} &= 0 \\ \mathrm{e}^x(x-1) &= 0 \\ x-1 &= 0 \\ x &= 1 \\ y &= \frac{\mathrm{e}^1}{2(1)} \\ &= \frac{\mathrm{e}}{2} \end{aligned}$$

32. $$\frac{\mathrm{d}R}{\mathrm{d}x} = 10\,000 \times -0.5\mathrm{e}^{-0.5x} = -5\,000\mathrm{e}^{-0.5x} \text{ \$/wk}$$

Exercise 7C

1. $$\frac{dy}{dx} = \frac{1}{x}$$

2. $\log_e 2x = \log_e 2 + \log_e x$
$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

3. $$\frac{dy}{dx} = 10x + \frac{1}{x}$$

4. $$\frac{dy}{dx} = 1 + e^x + \frac{1}{x}$$

5. Using the chain rule,
$$\frac{dy}{dx} = \frac{1}{3x+2} \times 3 = \frac{3}{3x+2}$$

6. $\log_e x^2 = 2\log_e x$
$$\frac{dy}{dx} = \frac{2}{x}$$

7. $\log_e(\sqrt[3]{x}) = \frac{1}{3}\log_e x$
$$\frac{dy}{dx} = \frac{1}{3x}$$

8. $\log_e(3\sqrt{x}) = \log_e 3 + \log_e \sqrt{x} = \log_e 3 + \frac{1}{2}\log_e x$
$$\frac{dy}{dx} = \frac{1}{2x}$$

9. $\log_e \frac{x}{5} = \log_e x - \log_e 5$
$$\frac{dy}{dx} = \frac{1}{x}$$

10. Using the chain rule with $u = x(x+3)$,
$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x(x+3)}\left[(x+3)+x\right] \\ &= \frac{2x+3}{x(x+3)}\end{aligned}$$

11. Using the chain rule with $u = x^2 + x - 12$,
$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x^2+x-12}(2x+1) \\ &= \frac{2x+1}{x^2+x-12}\end{aligned}$$

12. Using the product rule:
$$\begin{aligned}\frac{dy}{dx} &= \log_e(x) + x\frac{1}{x} \\ &= \log_e(x) + 1\end{aligned}$$

13. Using the chain rule with $u = \log_e x$,
$$\begin{aligned}\frac{dy}{dx} &= 3(\log_e x)^2\frac{1}{x} \\ &= \frac{3(\log_e x)^2}{x}\end{aligned}$$

14. Using the chain rule with $u = \frac{1}{x}$,
$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\frac{1}{x}}\left(-\frac{1}{x^2}\right) \\ &= x\left(-\frac{1}{x^2}\right) \\ &= -\frac{1}{x}\end{aligned}$$

The above works just fine, but it's simpler to use a log law first:
$$\begin{aligned}y &= \log_e \frac{1}{x} \\ &= -\log_e x \\ \frac{dy}{dx} &= -\frac{1}{x}\end{aligned}$$

It's not uncommon for there to be more than one way to do a problem, and it's similarly not uncommon to realise the simple approach only after you've done the more complicated one.

15. Using the chain rule with $u = \log_e x$,
$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{(\log_e x)^2} \times \frac{1}{x} \\ &= -\frac{1}{x(\log_e x)^2}\end{aligned}$$

16. Using the product rule,
$$\begin{aligned}\frac{dy}{dx} &= e^x \log_e x + e^x \frac{1}{x} \\ &= e^x\left(\log_e x + \frac{1}{x}\right)\end{aligned}$$

17. Using the quotient rule,
$$\begin{aligned}\frac{dy}{dx} &= \frac{(\frac{1}{x})(x) - (\log_e x)(1)}{x^2} \\ &= \frac{1-\log_e x}{x^2}\end{aligned}$$

18. Using the chain rule with $u = 1 + \log_e x$,
$$\begin{aligned}\frac{dy}{dx} &= 3(1+\log_e x)^2(\frac{1}{x}) \\ &= \frac{3(1+\log_e x)^2}{x}\end{aligned}$$

19. Using the chain rule with $u = 3x^2 + 16x + 15$. Note that we can find $\frac{du}{dx}$ using the product rule, but it's probably simpler to simply expand it: $u = x^3 + 8x^2 + 15x$.
$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x(x+5)(x+3)}(3x^2+16x+15) \\ &= \frac{3x^2+16x+15}{x(x+5)(x+3)}\end{aligned}$$

Again, however, this is simpler if we use log laws first:

$$\begin{aligned}y &= \log_e [x(x+5)(x+3)]\\ &= \log_e x + \log_e(x+5) + \log_e(x+3)\\ \frac{dy}{dx} &= \frac{1}{x} + \frac{1}{x+5} + \frac{1}{x+3}\end{aligned}$$

Students are encouraged to convince themselves that these two solutions are equivalent.

20. First simplify using log laws ...

$$\begin{aligned}y &= \log_e \frac{x+1}{x+3}\\ &= \log_e(x+1) - \log_e(x+3)\\ \frac{dy}{dx} &= \frac{1}{x+1} - \frac{1}{x+3}\end{aligned}$$

21. First simplify using log laws then use the chain rule with $u = x^2 + 5$:

$$\begin{aligned}y &= \log_e \left[(x^2+5)^4\right]\\ &= 4\log_e(x^2+5)\\ \frac{dy}{dx} &= 4\left(\frac{1}{x^2+5}\right)(2x)\\ &= \frac{8x}{x^2+5}\end{aligned}$$

22. Take as far as we can with log laws first, then differentiation is trivial.

$$\begin{aligned}y &= \log_e \frac{x}{x^2-1}\\ &= \log_e x - \log_e(x^2-1)\\ &= \log_e x - \log_e[(x-1)(x+1)]\\ &= \log_e x - \log_e(x-1) - \log_e(x+1)\\ \frac{dy}{dx} &= \frac{1}{x} - \frac{1}{x-1} - \frac{1}{x+1}\end{aligned}$$

23.
$$\begin{aligned}y &= \log_e \frac{(x+2)^3}{x-2}\\ &= \log_e[(x+2)^3] - \log_e(x-2)\\ &= 3\log_e(x+2) - \log_e(x-2)\\ \frac{dy}{dx} &= \frac{3}{x+2} - \frac{1}{x-2}\end{aligned}$$

24. $\frac{dy}{dx} = \frac{1}{x}$ so when $x = 1$, $\frac{dy}{dx} = \frac{1}{1} = 1$.

25. $\frac{dy}{dx} = \log_e(x)+1$ (see question 12) so when $x = e^2$,

$$\begin{aligned}\frac{dy}{dx} &= \log_e(e^2) + 1\\ &= 2+1\\ &= 3\end{aligned}$$

26. $y' = 6x + \frac{1}{x}$ so when $x = 1$, $y' = 6 \times 1 + \frac{1}{1} = 7$.

27. $y' = \frac{-2(1-\log_e x)}{x^2}$ (compare question 17) so when $x = 1$,

$$\begin{aligned}y' &= \frac{-2(1-\log_e(1))}{(1)^2}\\ &= \frac{-2(1-0)}{1}\\ &= -2\end{aligned}$$

28. $y' = \frac{1}{x}$ so

$$\begin{aligned}\frac{1}{x} &= 0.25\\ x &= 4\\ y &= \ln 4\end{aligned}$$

The coordinates of the one point with a gradient of 0.25 are $(4, \ln 4)$.

29. $y' = \frac{2}{x}$ (see question 6) so

$$\begin{aligned}\frac{2}{x} &= 4\\ x &= 0.5\\ y &= \ln 0.25\end{aligned}$$

The coordinates of the one point with a gradient of 4 are $(0.5, \ln 0.25)$.

30. Using the chain rule we obtain $y' = \frac{6}{6x-5}$ so

$$\begin{aligned}\frac{6}{6x-5} &= 0.24\\ \frac{1}{6x-5} &= 0.04\\ 6x-5 &= \frac{1}{0.04}\\ &= 25\\ x &= 5\\ y &= \ln(6(5)-5)\\ &= \ln 25\end{aligned}$$

The coordinates of the one point with a gradient of 0.24 are $(5, \ln 25)$.

31. Differentiating:

$$\begin{aligned}
y &= \ln x + \ln(x+3) \\
y' &= \frac{1}{x} + \frac{1}{x+3} \\
\frac{1}{x} + \frac{1}{x+3} &= 0.5 \\
(x+3) + x &= 0.5x(x+3) \\
2x + 3 &= 0.5(x^2 + 3x) \\
4x + 6 &= x^2 + 3x \\
x^2 + 3x - 4x - 6 &= 0 \\
x^2 - x - 6 &= 0 \\
(x-3)(x+2) &= 0 \\
\text{either} \quad x &= 3 \\
y &= \ln[3(3+3)] \\
&= \ln 18 \\
\text{or} \quad x &= -2 \\
y &= \ln[-2(-2+3)] \\
&= \ln(-2)
\end{aligned}$$

But $\ln(-2)$ is undefined so we have only one point with gradient 0.5 having coordinates $(3, \ln 18)$.

32. $y' = \frac{1}{x}$ so the gradient of the curve at $(1, 0)$ is $m = \frac{1}{1} = 1$, then the tangent line is

$$\begin{aligned}
(y - y_0) &= m(x - x_0) \\
y - 0 &= 1(x - 1) \\
y &= x - 1
\end{aligned}$$

33. The gradient is $m = \frac{1}{\mathrm{e}}$ so the tangent line is

$$\begin{aligned}
(y - y_0) &= m(x - x_0) \\
y - 1 &= \frac{1}{\mathrm{e}}(x - \mathrm{e}) \\
&= \frac{x}{\mathrm{e}} - 1 \\
y &= \frac{x}{\mathrm{e}}
\end{aligned}$$

Miscellaneous Exercise 7

1. (a) It's a sine curve with amplitude 1 and period of 180° so

$$y = 1 \sin 2x$$

(b) It's a cosine curve with amplitude 2 and period of 360° so

$$y = 2 \cos x$$

(c) It's an inverted sine with amplitude 2 and period 360° so

$$y = -2 \sin x$$

2. (a) $\mathrm{f}(x)$ is not differentiable at $x = a$ because it has a different gradient as we approach a from the left than when we approach from the right.

$\mathrm{f}(x)$ is not differentiable at $x = c$ because it is not continuous at that point.

$\mathrm{f}(x)$ is differentiable everywhere else (because linear and quadratic functions are always differentiable).

(b) $\mathrm{f}'(x)$ has a constant positive value for $x < a$, is undefined at $x = a$, is linear with positive gradient for $a < x < c$, passing through 0 when $x = b$, is undefined for $x = c$ and is zero for $x > c$.

See Sadler for a sketch of the graph.

3. Starting with the R.H.S.

$$\begin{aligned}
\frac{2\tan\theta}{\tan^2\theta + 1} &= \frac{2\frac{\sin\theta}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta} + 1} \\
&= \frac{2\frac{\sin\theta}{\cos\theta}}{\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}} \\
&= \frac{2\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}} \\
&= 2\frac{\sin\theta}{\cos\theta} \times \cos^2\theta \\
&= 2\sin\theta\cos\theta \\
&= \sin 2\theta \\
&= \text{L.H.S.}
\end{aligned}$$

□

4. (a) Chain rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 4(3x^2 + 5)^3(6x) \\
&= 24x(3x^2 + 5)^3
\end{aligned}$$

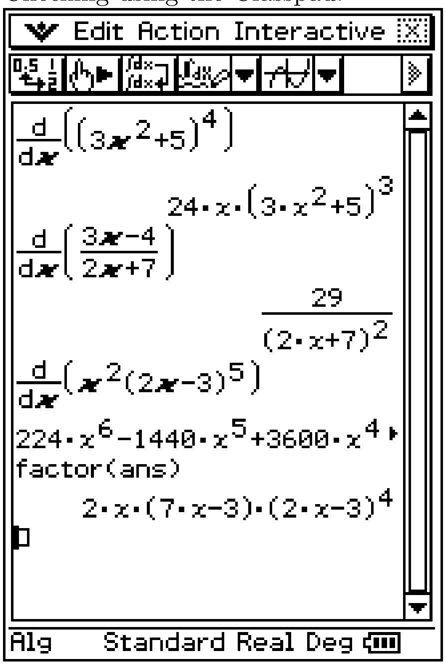
(b) Quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{3(2x+7)-2(3x-4)}{(2x+7)^2}\\ &= \frac{6x+21-6x+8}{(2x+7)^2}\\ &= \frac{29}{(2x+7)^2}\end{aligned}$$

(c) Product and chain rules:

$$\begin{aligned}\frac{dy}{dx} &= 2x(2x-3)^5 + x^2[5(2x-3)^4](2)\\ &= 2x(2x-3)^5 + 10x^2(2x-3)^4\\ &= 2x(2x-3)^4(2x-3\ +5x)\\ &= 2x(2x-3)^4(7x-3)\end{aligned}$$

Checking using the Classpad:



5. (a)
$$\begin{aligned}3x^2+11x-4 &= 3x^2+12x-x-4\\ &= 3x(x+4)-(x+4)\\ &= (x+4)(3x-1)\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to -4}\frac{3x^2+11x-4}{x+4} &= \lim_{x\to -4}\frac{(x+4)(3x-1)}{x+4}\\ &= \lim_{x\to -4}(3x-1)\\ &= 3(-4)-1\\ &= -13\end{aligned}$$

6. Continuity at $x=3$:

$$\begin{aligned}\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^+} f(x)\\ \lim_{x\to 3^-}(2x+5) &= \lim_{x\to 3^+}(ax+2)\\ 11 &= 3a+2\\ a &= 3\end{aligned}$$

Continuity at $x=10$:

$$\begin{aligned}\lim_{x\to 10^-} f(x) &= \lim_{x\to 10^+} f(x)\\ \lim_{x\to 10^-}(3x+2) &= \lim_{x\to 10^+}(4x+b)\\ 32 &= 40+b\\ b &= -8\end{aligned}$$

7. a represents the $x-$value where the function is discontinuous, i.e. where the denominator goes to zero:

$$a = 3$$

b represents the $y-$value that the function approaches when $x\to\infty$. Using the principle of dominant powers of x,

$$b = \lim_{x\to\infty}\frac{2x+12}{x-3} = \frac{2x}{x} = 2$$

Point C is the $x-$intercept, i.e.

$$\begin{aligned}\frac{2x+12}{x-3} &= 0\\ 2x+12 &= 0\\ x &= -6\end{aligned}$$

so point C has coordinates $(-6,0)$.

Point D is the $y-$intercept, i.e.

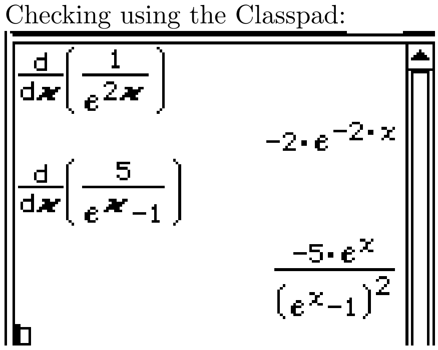
$$\begin{aligned}y &= \frac{2(0)+12}{(0)-3}\\ &= -4\end{aligned}$$

so point D has coordinates $(0,-4)$.

8. (a)
$$\begin{aligned}\frac{d}{dx}\frac{1}{e^{2x}} &= \frac{d}{dx}e^{-2x}\\ &= -2e^{-2x}\\ &= -\frac{2}{e^{2x}}\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dx}\frac{5}{e^x-1} &= \frac{0-5e^x}{(e^x-1)^2}\\ &= -\frac{5e^x}{(e^x-1)^2}\end{aligned}$$

Checking using the Classpad:



9.
$$\begin{aligned}\lim_{x\to 2^-} f(x) &= 3(2)\\ &= 6\\ \lim_{x\to 2^+} f(x) &= (2)+2\\ &= 4\end{aligned}$$

The limit does not exist.

10.
$$\begin{aligned}\lim_{x\to 1^-} \mathrm{g}(x) &= 1+2\\ &= 3\\ \lim_{x\to 1^+} \mathrm{g}(x) &= 3(1)\\ &= 3\\ \therefore \quad \lim_{x\to 1} \mathrm{g}(x) &= 3\end{aligned}$$

11. (a)
$$\begin{aligned}\lim_{x\to 0^-} \mathrm{f}(x) &= \frac{2(0)+3}{0-1}\\ &= -3\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to 0^+} \mathrm{f}(x) &= \frac{0-6}{0+2}\\ &= -3\end{aligned}$$

(c)
$$\lim_{x\to 0} \mathrm{f}(x) = -3$$

(d)
$$\begin{aligned}\lim_{x\to -\infty} \mathrm{f}(x) &= \lim_{x\to -\infty} \frac{2x+3}{x-1}\\ &= 2\end{aligned}$$

(e)
$$\begin{aligned}\lim_{x\to \infty} \mathrm{f}(x) &= \lim_{x\to -\infty} \frac{x-6}{x+1}\\ &= 1\end{aligned}$$

12. (a) The absolute value function is continuous, so the limit is equal to the value of the function:

$$\lim_{x\to 0}(|x|+3) = |0|+3 = 3$$

(b) The limit does not exist. The limit from the left is -1 while from the right it is 1.

(c) The function is continuous at $x = 0$ (its only discontinuity is at $x = 3$) so

$$\lim_{x\to 0} = \frac{|0-3|}{0-3} = -1$$

(d) The limit does not exist. The limit from the left is -1 while from the right it is 1.

13. We need only concern ourselves with continuity and differentiabilty at $x = 2$ as everywhere else we are dealing with polynomial functions.

For continuity,

$$\begin{aligned}-23(-2) - 28 &= a(-2) + b((-2)^2)\\ -2a + 4b &= 18\\ -a + 2b &= 9\end{aligned}$$

For differentiability,

$$\begin{aligned}-23 &= a + 2b(-2)\\ a - 4b &= -23\end{aligned}$$

Solving simultaneously

$$\begin{aligned}-2b &= -14\\ b &= 7\\ -a + 2b &= 9\\ -a + 14 &= 9\\ a &= 5\end{aligned}$$

14.
$$\begin{aligned}z - 2 + 7\mathrm{i} &= \pm 5\mathrm{i}\\ x &= 2 - 12\mathrm{i}\\ \text{or} \quad x &= 2 - 2\mathrm{i}\end{aligned}$$

15.
$$\begin{aligned}\sin 105^\circ &= \sin(60^\circ + 45^\circ)\\ &= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ\\ &= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}(\sqrt{3}+1)}{4}\end{aligned}$$

16. Refer to the sketches in Sadler's solutions.

(a) The gradient is positive everywhere and tends towards zero as $x \to \pm\infty$. As we approach $x = a$ from the left or the right the gradient increases without bound.

(b) Here the gradient is negative everywhere and tends towards zero as $x \to \pm\infty$. As we approach $x = a$ from the left or the right the gradient decreases without bound.

(c) The gradient is positive for $x < a$, as $x \to -\infty$ so $\mathrm{h}'(x) \to 0$ and as $x \to a^-$ so $\mathrm{h}'(x) \to \infty$.
The gradient is negative for $x > a$, as $x \to \infty$ so $\mathrm{h}'(x) \to 0$ and as $x \to a^+$ so $\mathrm{h}'(x) \to -\infty$.

17. The position of the boat after 3 hours (i.e. at noon) is $3(12\mathbf{i} + 4\mathbf{j}) = (36\mathbf{i} + 12\mathbf{j})$km. The displacement from there to the harbour at A is

$$(42\mathbf{i} + 9\mathbf{j}) - (36\mathbf{i} + 12\mathbf{j}) = (6\mathbf{i} - 3\mathbf{j})\text{km}$$

Let the boat's velocity be $\mathbf{v} = (a\mathbf{i} + b\mathbf{j})$km/h. To steam directly to A, the vector sum of this velocity and that of the wind and current must be a scalar multiple of this displacement:

$$(a\mathbf{i} + b\mathbf{j}) + (6\mathbf{i} + 2\mathbf{j}) = k(6\mathbf{i} - 3\mathbf{j})$$

(where $k = \frac{1}{t}$ for t the time in hours)

$$\begin{aligned}(a\mathbf{i} + b\mathbf{j}) &= k(6\mathbf{i} - 3\mathbf{j}) - (6\mathbf{i} + 2\mathbf{j})\\ a &= 6k - 6\\ b &= -3k - 2\end{aligned}$$

From the boats speed of 10km/h,

$$\begin{aligned}a^2 + b^2 &= 10^2\\ (6k-6)^2 + (-3k-2)^2 &= 100\\ 36k^2 - 72k + 36 + 9k^2 + 12k + 4 - 100 &= 0\\ 45k^2 - 60k - 60 &= 0\\ 3k^2 - 4k - 4 &= 0\\ 3k^2 - 6k + 2k - 4 &= 0\\ 3k(k-2) + 2(k-2) &= 0\\ (k-2)(3k+2) &= 0\\ k &= 2\end{aligned}$$

(we can ignore the other root because it gives negative time)

$$\begin{aligned} a &= 6k - 6 \\ &= 6 \\ b &= -3k - 2 \\ &= -8 \\ t &= \frac{1}{2} \end{aligned}$$

The velocity of the boat should be set to $(6\mathbf{i} - 8\mathbf{j})$km/h. The boat will arrive after half an hour, i.e. at 12:30pm.