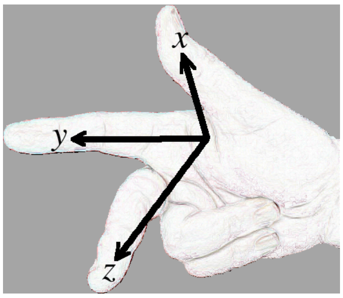
Chapter 3

Exercise 3A

1. For students with limited experience with hardware, the right hand screw rule explained in Sadler is not a very helpful mnemonic. An alternative method of remembering the relationships between axes in three dimensions is the "Right Hand Rule". In this rule, the x, y and z axes correspond to thumb, index finger and middle finger respectively as illustrated below.



2. (a) $\mathbf{a}+\mathbf{b} = (2+3)\mathbf{i} + (6+8)\mathbf{j} + (3-1)\mathbf{k}$
$= 5\mathbf{i} + 14\mathbf{j} + 2\mathbf{k}$

(b) $\mathbf{a}-\mathbf{b} = (2-3)\mathbf{i} + (6-8)\mathbf{j} + (3--1)\mathbf{k}$
$= -\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

(c) $2\mathbf{a}+\mathbf{b} = (2\times 2+3)\mathbf{i} + (2\times 6+8)\mathbf{j}$
$+ (2\times 3-1)\mathbf{k}$
$= 7\mathbf{i} + 20\mathbf{j} + 5\mathbf{k}$

(d) $2(\mathbf{a}+\mathbf{b}) = 2(5\mathbf{i} + 14\mathbf{j} + 2\mathbf{k})$
$= 10\mathbf{i} + 28\mathbf{j} + 4\mathbf{k}$

(e) $\mathbf{a}\cdot\mathbf{b} = 2\times 3 + 6\times 8 + 3\times(-1)$
$= 51$

(f) $\mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b}$
$= 51$

(g) $|\mathbf{a}| = \sqrt{2^2+6^2+3^2}$
$= \sqrt{49}$
$= 7$

(h) $|\mathbf{a}+\mathbf{b}| = \sqrt{5^2+14^2+2^2}$
$= \sqrt{225}$
$= 15$

3. (a) $\mathbf{c}+\mathbf{d} = \begin{pmatrix} -1+2 \\ 4+0 \\ 3+4 \end{pmatrix}$
$= \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

(b) $\mathbf{c}-\mathbf{d} = \begin{pmatrix} -1-2 \\ 4-0 \\ 3-4 \end{pmatrix}$
$= \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$

(c) $2\mathbf{c}+\mathbf{d} = \begin{pmatrix} 2\times -1+2 \\ 2\times 4+0 \\ 2\times 3+4 \end{pmatrix}$
$= \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$

(d) $2(\mathbf{c}+\mathbf{d}) = 2\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$
$= \begin{pmatrix} 2 \\ 8 \\ 14 \end{pmatrix}$

(e) $\mathbf{c}\cdot\mathbf{d} = -1\times 2 + 4\times 0 + 3\times 4$
$= 10$

(f) $\mathbf{d}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{d}$
$= 10$

(g) $|\mathbf{c}| = \sqrt{(-1)^2+4^2+3^2}$
$= \sqrt{26}$

(h) $|\mathbf{c}+\mathbf{d}| = \sqrt{1^2+4^2+7^2}$
$= \sqrt{66}$

4. (a) $\mathbf{e}-\mathbf{f} = \langle 1--1, 4-2, -3-0\rangle$
$= \langle 2, 2, -3\rangle$

(b) $\mathbf{e}-2\mathbf{f} = \langle 1-2\times -1, 4-2\times 2, -3-2\times 0\rangle$
$= \langle 3, 0, -3\rangle$

(c) $2\mathbf{e}+\mathbf{f} = \langle 2\times 1+-1, 2\times 4+2, 2\times -3+0\rangle$
$= \langle 1, 10, -6\rangle$

(d) $\mathbf{e}+\mathbf{f} = \langle 1+-1, 4+2, -3+0\rangle$
$= \langle 0, 6, -3\rangle$

(e) $\mathbf{e}\cdot\mathbf{f} = 1\times -1 + 4\times 2 + -3\times 0$
$= 7$

(f) $(2\mathbf{e})\cdot(3\mathbf{f}) = 2\times 3\times \mathbf{e}\cdot\mathbf{f}$
$= 6\times 7$
$= 42$

(g) $(\mathbf{e}-\mathbf{f})\cdot(\mathbf{e}-\mathbf{f}) = 2^2+2^2+(-3)^2$
$= 17$

(h) $|\mathbf{e}-\mathbf{f}| = \sqrt{2^2+2^2+(-3)^2}$
$= \sqrt{17}$

5. (a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= (3\mathbf{i}+2\mathbf{j}+\mathbf{k}) - (2\mathbf{i}+3\mathbf{j}+-4\mathbf{k})$
$= \mathbf{i}-\mathbf{j}+5\mathbf{k}$

(b) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$
$= (-5\mathbf{i}-\mathbf{j}+\mathbf{k}) - (3\mathbf{i}+2\mathbf{j}+\mathbf{k})$
$= -8\mathbf{i}-3\mathbf{j}$

(c) $\overrightarrow{\mathrm{CA}} = \overrightarrow{\mathrm{OA}} - \overrightarrow{\mathrm{OC}}$
$= (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) - (-5\mathbf{i} - \mathbf{j} + \mathbf{k})$
$= 7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$

(d) $\overrightarrow{\mathrm{AC}} = -\overrightarrow{\mathrm{AC}}$
$= -(7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$
$= -7\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

6. (a) $\mathbf{p} + \mathbf{q} = \begin{pmatrix} 3+4 \\ 2-1 \\ 1+3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix}$

(b) $\mathbf{q} + \mathbf{r} = \begin{pmatrix} 4+2 \\ -1+0 \\ 3+1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix}$

(c) $(\mathbf{p} + \mathbf{q}) \cdot (\mathbf{q} + \mathbf{r}) = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix}$
$= 7 \times 6 + 1 \times -1 + 4 \times 4$
$= 57$

7. (a) $|\mathbf{u}| = \sqrt{3^2 + (-2)^2 + 6^2}$
$= 7$

(b) $|\mathbf{v}| = \sqrt{2^2 + 14^2 + 5^2}$
$= 15$

(c) $\mathbf{u} \cdot \mathbf{v} = 3 \times 2 + -2 \times 14 + 6 \times 5$
$= 8$

(d) $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$
$8 = 7 \times 15 \cos\theta$
$\theta = \cos^{-1} \frac{8}{105}$
$= 85.6°$

8. $\cos\angle\mathrm{AOB} = \frac{\overrightarrow{\mathrm{OA}} \cdot \overrightarrow{\mathrm{OB}}}{|\overrightarrow{\mathrm{OA}}||\overrightarrow{\mathrm{OB}}|}$
$= \frac{1 \times 2 + 1 \times -1 + -1 \times 2}{\sqrt{1^2 + 1^2 + (-1)^2}\sqrt{2^2 + (-1)^2 + 2^2}}$
$= \frac{-1}{3\sqrt{3}}$
$\angle\mathrm{AOB} = \cos^{-1} \frac{-1}{3\sqrt{3}}$
$= 101°$

9. $\cos\theta = \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$
$= \frac{-1 \times -1 + 2 \times 1 + 1 \times -2}{\sqrt{(-1)^2 + 2^2 + 1^2}\sqrt{(-1)^2 + 1^2 + (-2)^2}}$
$= \frac{1}{6}$
$\theta = \cos^{-1} \frac{1}{6}$
$= 80°$

10. $\cos\theta = \frac{\mathbf{s} \cdot \mathbf{t}}{|\mathbf{s}||\mathbf{t}|}$
$= \frac{2 \times 3 + 1 \times 0 + -1 \times 3}{\sqrt{2^2 + 1^2 + (-1)^2}\sqrt{3^2 + 0^2 + 3^2}}$
$= \frac{3}{\sqrt{6} \times 3\sqrt{2}}$
$= \frac{1}{\sqrt{6}\sqrt{2}}$
$\theta = \cos^{-1} \frac{1}{\sqrt{6}\sqrt{2}}$
$= 73°$

11. (a) Let $\mathbf{u}$ be a scalar multiple of $\mathbf{r}$ having unit magnitude.
$|\mathbf{r}| = \sqrt{2^2 + (-3)^2 + 6^2}$
$= 7$
$\therefore \quad \mathbf{u} = \frac{1}{7}\mathbf{r}$
$= \frac{1}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

(b) Let $\mathbf{v}$ be a scalar multiple of $\mathbf{r}$ having the same magnitude as $\mathbf{s}$. If it's a scalar multiple of $\mathbf{r}$ then it's also a scalar multiple of $\mathbf{u}$, so
$|\mathbf{s}| = \sqrt{3^2 + 4^2}$
$= 5$
$\therefore \quad \mathbf{v} = 5\mathbf{u}$
$= \frac{5}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

(c) Scale $\mathbf{s}$ by the ratio of the magnitudes of $\mathbf{r}$ and $\mathbf{s}$.
$\frac{|\mathbf{r}|}{|\mathbf{s}|}\mathbf{s} = \frac{7}{5}(3\mathbf{i} + 4\mathbf{k})$

(d) $\cos\theta = \frac{\mathbf{r} \cdot \mathbf{s}}{|\mathbf{r}||\mathbf{s}|}$
$= \frac{2 \times 3 - 3 \times 0 + 6 \times 4}{7 \times 5}$
$= \frac{30}{35}$
$= \frac{6}{7}$
$\theta = \cos^{-1} \frac{6}{7}$
$= 31°$

12. (a) Vectors are scalar multiples of each other (the second is double the first) so they are parallel.

(b) Not scalar multiples, so not parallel.
$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 3 - 2 - 3$
$= -2$
Dot product is not zero, so not perpendicular.

(c) Not scalar multiples, so not parallel.
$\langle 1, 3, -2 \rangle \cdot \langle -2, 3, 1 \rangle = -2 + 9 - 2$
$= 5$
Dot product is not zero, so not perpendicular.

(d) Not scalar multiples, so not parallel.

$$\begin{aligned}\langle 1,2,3\rangle \cdot \langle 3,3,-3\rangle &= 3+6-9\\ &= 0\end{aligned}$$

Dot product is zero, so vectors are perpendicular.

(e) Not scalar multiples, so not parallel.

$$\begin{aligned}\begin{pmatrix}3\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}5\\-7\\1\end{pmatrix} &= 15-14-1\\ &= 0\end{aligned}$$

Dot product is zero, so vectors are perpendicular.

(f) Not scalar multiples, so not parallel.

$$\begin{aligned}\begin{pmatrix}-2\\6\\8\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\4\end{pmatrix} &= -2-18+32\\ &= 12\end{aligned}$$

Dot product is not zero, so vectors are not perpendicular.

(g) Not scalar multiples, so not parallel.

$$\begin{aligned}\begin{pmatrix}3\\1\\5\end{pmatrix}\cdot\begin{pmatrix}6\\2\\-4\end{pmatrix} &= 18+2-20\\ &= 0\end{aligned}$$

Dot product is zero, so vectors are perpendicular.

13.
$$\begin{aligned}\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3 &= (5+3-2)\mathbf{i}+(10+5+3)\mathbf{j}\\ &\quad +(5+5-1)\mathbf{k}\\ &= (6\mathbf{i}+18\mathbf{j}+9\mathbf{k})\,\text{N}\\ |\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3| &= \sqrt{6^2+18^2+9^2}\\ &= 21\text{N}\end{aligned}$$

14.
$$\begin{aligned}\overrightarrow{\text{BA}} &= \overrightarrow{\text{OA}}-\overrightarrow{\text{OB}}\\ -\mathbf{i}+3\mathbf{j}+4\mathbf{k} &= (2\mathbf{i}+3\mathbf{j}-4\mathbf{k})-\overrightarrow{\text{OB}}\\ \overrightarrow{\text{OB}} &= (2\mathbf{i}+3\mathbf{j}-4\mathbf{k})-(-\mathbf{i}+3\mathbf{j}+4\mathbf{k})\\ &= 3\mathbf{i}-8\mathbf{k}\end{aligned}$$

The position vector of B is $3\mathbf{i}-8\mathbf{k}$.

15. $(\mathbf{a}+\mathbf{b})+(\mathbf{a}-\mathbf{b})=2\mathbf{a}$

$$\begin{aligned}\begin{pmatrix}7\\1\\2\end{pmatrix}+\begin{pmatrix}3\\3\\-4\end{pmatrix} &= \begin{pmatrix}10\\4\\-2\end{pmatrix}\\ \therefore\quad \mathbf{a} &= \frac{1}{2}\begin{pmatrix}10\\4\\-2\end{pmatrix}\\ &= \begin{pmatrix}5\\2\\-1\end{pmatrix}\end{aligned}$$

$(\mathbf{a}+\mathbf{b})-(\mathbf{a}-\mathbf{b})=2\mathbf{b}$

$$\begin{aligned}\begin{pmatrix}7\\1\\2\end{pmatrix}-\begin{pmatrix}3\\3\\-4\end{pmatrix} &= \begin{pmatrix}4\\-2\\6\end{pmatrix}\\ \therefore\quad \mathbf{b} &= \frac{1}{2}\begin{pmatrix}4\\-2\\6\end{pmatrix}\\ &= \begin{pmatrix}2\\-1\\3\end{pmatrix}\end{aligned}$$

16. $\mathbf{b}$ is parallel to $\mathbf{a}$ so they are scalar multiples, i.e. $\mathbf{b}=k\mathbf{a}$. By examining the $\mathbf{i}$ and $\mathbf{j}$ components we can see that $k=2$ so $p=2\times 1=2$.

$\mathbf{c}$ is perpendicular to $\mathbf{a}$ so they have zero dot product, i.e.

$$\begin{aligned}2\times 7+3q+1\times -2 &= 0\\ 12+3q &= 0\\ q &= -4\end{aligned}$$

$\mathbf{d}$ is perpendicular to $\mathbf{b}$, but $\mathbf{b}$ is parallel to $\mathbf{a}$ so $\mathbf{d}$ is perpendicular to $\mathbf{a}$.

$$\begin{aligned}\mathbf{d}\cdot\mathbf{a} &= 0\\ 3\times 2-4\times 3+r\times 1 &= 0\\ -6+r &= 0\\ r &= 6\end{aligned}$$

Note that even though both are perpendicular to $\mathbf{a}$, $\mathbf{c}$ is not parallel to $\mathbf{d}$ (as they would be in two dimensions).

17. (a) $(-4\mathbf{i}-4\mathbf{j}+11\mathbf{k})+(2\mathbf{i}+4\mathbf{j}-2\mathbf{k})=(-2\mathbf{i}+9\mathbf{k})\,\text{m}$

(b) $(-4\mathbf{i}-4\mathbf{j}+11\mathbf{k})+2\,(2\mathbf{i}+4\mathbf{j}-2\mathbf{k})=(4\mathbf{j}+7\mathbf{k})\,\text{m}$

(c)
$$\begin{aligned}(-4\mathbf{i}-4\mathbf{j}+11\mathbf{k})+3\,(2\mathbf{i}+4\mathbf{j}-2\mathbf{k}) &= (2\mathbf{i}+8\mathbf{j}+5\mathbf{k})\,\text{m}\\ |\,(2\mathbf{i}+8\mathbf{j}+5\mathbf{k})\,| &= \sqrt{2^2+8^2+5^2}\\ &= 9.6\text{m}\end{aligned}$$

(d)
$$\begin{aligned}|\,(-4\mathbf{i}-4\mathbf{j}+11\mathbf{k})+t\,(2\mathbf{i}+4\mathbf{j}-2\mathbf{k})\,| &= 15\\ |(-4+2t)\mathbf{i}+(-4+4t)\mathbf{j}+(11-2t)\mathbf{k}| &= 15\\ \sqrt{(-4+2t)^2+(-4+4t)^2+(11-2t)^2} &= 15\\ (-4+2t)^2+(-4+4t)^2+(11-2t)^2 &= 15^2\\ 24t^2-92t+153 &= 225\end{aligned}$$

$t=4.5\text{s}$ (ignoring the negative root)

18. A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{AC}$.

$$\begin{aligned}\overrightarrow{AB} &= (3\mathbf{i}+\mathbf{j}-4\mathbf{k})-(7\mathbf{i}+5\mathbf{j})\\ &= -4\mathbf{i}-4\mathbf{j}-4\mathbf{k}\\ \overrightarrow{AC} &= (2\mathbf{i}-5\mathbf{k})-(7\mathbf{i}+5\mathbf{j})\\ &= -5\mathbf{i}-5\mathbf{j}-5\mathbf{k}\\ \overrightarrow{AB} &= \frac{4}{5}\overrightarrow{AC}\end{aligned}$$

$\therefore$ A, B and C are collinear. □

19. Let P be the point that divides AB in the ratio 2:3.

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA}+\frac{2}{5}\overrightarrow{AB}\\ &= \begin{pmatrix}3\\4\\4\end{pmatrix}+\frac{2}{5}\left(\begin{pmatrix}-2\\9\\-1\end{pmatrix}-\begin{pmatrix}3\\4\\4\end{pmatrix}\right)\\ &= \begin{pmatrix}3\\4\\4\end{pmatrix}+\frac{2}{5}\begin{pmatrix}-5\\5\\-5\end{pmatrix}\\ &= \begin{pmatrix}3\\4\\4\end{pmatrix}+\begin{pmatrix}-2\\2\\-2\end{pmatrix}\\ &= \begin{pmatrix}1\\6\\2\end{pmatrix}\end{aligned}$$

20. $$\begin{aligned}\overrightarrow{AB} &= (4\mathbf{i}-\mathbf{j}+\mathbf{k})-(3\mathbf{i}+2\mathbf{j}-\mathbf{k})\\ &= \mathbf{i}-3\mathbf{j}+2\mathbf{k}\\ \overrightarrow{OP} &= \overrightarrow{OB}+\overrightarrow{BP}\\ &= \overrightarrow{OB}+\overrightarrow{AB}\\ &= (4\mathbf{i}-\mathbf{j}+\mathbf{k})+(\mathbf{i}-3\mathbf{j}+2\mathbf{k})\\ &= 5\mathbf{i}-4\mathbf{j}+3\mathbf{k}\end{aligned}$$

21. $$\begin{aligned}\overrightarrow{AB} &= (9\mathbf{i}+6\mathbf{j}-9\mathbf{k})-(5\mathbf{i}-2\mathbf{j}+3\mathbf{k})\\ &= 4\mathbf{i}+8\mathbf{j}-12\mathbf{k}\\ \overrightarrow{OP} &= \overrightarrow{OA}+\frac{3}{4}\overrightarrow{AB}\\ &= (5\mathbf{i}-2\mathbf{j}+3\mathbf{k})+\frac{3}{4}(4\mathbf{i}+8\mathbf{j}-12\mathbf{k})\\ &= (5\mathbf{i}-2\mathbf{j}+3\mathbf{k})+(3\mathbf{i}+6\mathbf{j}-9\mathbf{k})\\ &= 8\mathbf{i}+4\mathbf{j}-6\mathbf{k}\end{aligned}$$

22. $$\begin{aligned}\overrightarrow{AB} &= (3\mathbf{k})-(2\mathbf{i}+3\mathbf{j}+2\mathbf{k})\\ &= -2\mathbf{i}-3\mathbf{j}+\mathbf{k}\\ \overrightarrow{AC} &= (4\mathbf{i}-3\mathbf{j}+2\mathbf{k})-(2\mathbf{i}+3\mathbf{j}+2\mathbf{k})\\ &= 2\mathbf{i}-6\mathbf{j}\\ \overrightarrow{BC} &= (4\mathbf{i}-3\mathbf{j}+2\mathbf{k})-(3\mathbf{k})\\ &= 4\mathbf{i}-3\mathbf{j}-\mathbf{k}\\ \overrightarrow{AB}\cdot\overrightarrow{AC} &= -4+18+0\\ &= 14\\ \overrightarrow{AB}\cdot\overrightarrow{BC} &= -8+9-1\\ &= 0\end{aligned}$$

$\therefore$ AB $\perp$ BC and $\triangle$ABC is right angled at B. □

23. Let α be the angle $\mathbf{a}$ makes with the $x-$axis.

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|}\\ &= \frac{(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(\mathbf{i})}{\sqrt{2^2+3^2+(-1)^2}\times 1}\\ &= \frac{2}{\sqrt{14}}\\ \alpha &= \cos^{-1}\frac{2}{\sqrt{14}}\\ &= 57.7°\end{aligned}$$

Let β be the angle $\mathbf{a}$ makes with the $y-$axis.

$$\begin{aligned}\cos\beta &= \frac{\mathbf{a}\cdot\mathbf{j}}{|\mathbf{a}||\mathbf{j}|}\\ &= \frac{(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(\mathbf{j})}{\sqrt{14}}\\ &= \frac{3}{\sqrt{14}}\\ \beta &= \cos^{-1}\frac{3}{\sqrt{14}}\\ &= 36.7°\end{aligned}$$

Let γ be the angle $\mathbf{a}$ makes with the $z-$axis.

$$\begin{aligned}\cos\gamma &= \frac{\mathbf{a}\cdot\mathbf{k}}{|\mathbf{a}||\mathbf{k}|}\\ &= \frac{(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(\mathbf{k})}{\sqrt{14}}\\ &= \frac{-1}{\sqrt{14}}\\ \gamma &= \cos^{-1}\frac{-1}{\sqrt{14}}\\ &= 105.5°\end{aligned}$$

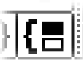
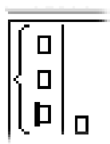
We want the acute angle, so we need the supplementary angle:
$180-105.5 = 74.5°$

24. **Vector d:**

$$\begin{aligned}\mathbf{d} &= \lambda\mathbf{a}+\mu\mathbf{b}+\eta\mathbf{c}\\ 7\mathbf{i}-5\mathbf{j}+10\mathbf{k} &= \lambda(\mathbf{i}-2\mathbf{j}+3\mathbf{k})\\ &\quad+\mu(2\mathbf{i}+\mathbf{j}-\mathbf{k})\\ &\quad+\eta(4\mathbf{i}-\mathbf{j}+3\mathbf{k})\end{aligned}$$

Equating $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ components gives three equations in three unknowns:

$$\begin{aligned}\lambda\quad +2\mu\quad +4\eta &= 7\\ -2\lambda\quad +\mu\quad -\eta &= -5\\ 3\lambda\quad -\mu\quad +3\eta &= 10\end{aligned}$$

Solving this using the Classpad. On the 2D tab, tap on . Tap the same icon a second time to expand to three lines . Fill in the three equations. Use x, y, z in place of the Greek letters:

```
⎧ x+2y+4z=7
⎨ -2x+y-z=-5
⎩ 3x-y+3z=10 | x,y,z
        {x=1,y=-1,z=2}
```

$\mathbf{d} = \mathbf{a} - \mathbf{b} + 2\mathbf{c}$

Vector e:

$$\begin{aligned} \mathbf{e} &= \lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c} \\ \mathbf{i} - 5\mathbf{j} + 8\mathbf{k} &= \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &\quad + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ &\quad + \eta(4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \end{aligned}$$

Equating $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ components gives three equations in three unknowns:

$$\begin{aligned} \lambda \quad +2\mu \quad +4\eta &= 1 \\ -2\lambda \quad +\mu \quad -\eta &= -5 \\ 3\lambda \quad -\mu \quad +3\eta &= 8 \end{aligned}$$

Solving this using the Classpad gives $\lambda = 1, \mu = -2, \eta = 1$.

$\mathbf{e} = \mathbf{a} - 2\mathbf{b} + \mathbf{c}$

Vector f:

$$\begin{aligned} \mathbf{f} &= \lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c} \\ 2\mathbf{j} - 2\mathbf{k} &= \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &\quad + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ &\quad + \eta(4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \end{aligned}$$

Equating $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ components gives three equations in three unknowns:

$$\begin{aligned} \lambda \quad +2\mu \quad +4\eta &= 0 \\ -2\lambda \quad +\mu \quad -\eta &= 2 \\ 3\lambda \quad -\mu \quad +3\eta &= -2 \end{aligned}$$

Solving this using the Classpad gives $\lambda = -2, \mu = -1, \eta = 1$.

$\mathbf{f} = -2\mathbf{a} - \mathbf{b} + \mathbf{c}$

25. (a)
$$\begin{aligned} \overrightarrow{\mathrm{DC}} &= (10\mathbf{i})\,\mathrm{cm} \\ \overrightarrow{\mathrm{DB}} &= (10\mathbf{i} + 4\mathbf{k})\,\mathrm{cm} \\ \overrightarrow{\mathrm{DI}} &= (3\mathbf{j} + \mathbf{k})\,\mathrm{cm} \end{aligned}$$

(b)
$$\begin{aligned} \cos\angle\mathrm{IDB} &= \frac{\overrightarrow{\mathrm{DI}} \cdot \overrightarrow{\mathrm{DB}}}{|\overrightarrow{\mathrm{DI}}||\overrightarrow{\mathrm{DB}}|} \\ &= \frac{(3\mathbf{j} + \mathbf{k}) \cdot (10\mathbf{i} + 4\mathbf{k})}{|(3\mathbf{j} + \mathbf{k})||(10\mathbf{i} + 4\mathbf{k})|} \\ &= \frac{0 + 0 + 4}{\sqrt{0^2 + 3^2 + 1^2}\sqrt{10^2 + 0^2 + 4^2}} \\ &= \frac{4}{\sqrt{10}\sqrt{116}} \\ \angle\mathrm{IDB} &= 83° \end{aligned}$$

26. (a)
$$\begin{aligned} \overrightarrow{\mathrm{AO}} &= (\mathbf{0}) - (4\mathbf{i} + 2\mathbf{j}) \\ &= (-4\mathbf{i} - 2\mathbf{j}) \\ |\overrightarrow{\mathrm{AO}}| &= \sqrt{(-4)^2 + (-2)^2 + 0^2} \\ &= \sqrt{20} \\ \overrightarrow{\mathrm{AE}} &= (8\mathbf{k}) - (4\mathbf{i} + 2\mathbf{j}) \\ &= (-4\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}) \\ |\overrightarrow{\mathrm{AE}}| &= \sqrt{(-4)^2 + (-2)^2 + 8^2} \\ &= \sqrt{84} \\ \cos\angle\mathrm{OAE} &= \frac{\overrightarrow{\mathrm{AO}} \cdot \overrightarrow{\mathrm{AE}}}{|\overrightarrow{\mathrm{AO}}||\overrightarrow{\mathrm{AE}}|} \\ &= \frac{16 + 4 + 0}{\sqrt{20}\sqrt{84}} \\ &= \frac{20}{\sqrt{20}\sqrt{84}} \\ \angle\mathrm{OAE} &= 60.8° \end{aligned}$$

(b)
$$\begin{aligned} \overrightarrow{\mathrm{DB}} &= (-4\mathbf{i} + 2\mathbf{j}) - (4\mathbf{i} - 2\mathbf{j}) \\ &= (-8\mathbf{i} + 4\mathbf{j}) \\ |\overrightarrow{\mathrm{DB}}| &= \sqrt{(-8)^2 + 4^2 + 0^2} \\ &= \sqrt{80} \\ \cos\theta &= \frac{\overrightarrow{\mathrm{AE}} \cdot \overrightarrow{\mathrm{DB}}}{|\overrightarrow{\mathrm{AE}}||\overrightarrow{\mathrm{DB}}|} \\ &= \frac{32 - 8 + 0}{\sqrt{84}\sqrt{80}} \\ &= \frac{24}{\sqrt{84}\sqrt{80}} \\ \theta &= 73.0° \end{aligned}$$

27. (a)
$$\begin{aligned} \overrightarrow{\mathrm{AB}} &= \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} \\ \overrightarrow{\mathrm{BC}} &= \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \\ \overrightarrow{\mathrm{AC}} &= \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} \end{aligned}$$

(b)
$$\begin{aligned} |\overrightarrow{\mathrm{AB}}| &= \sqrt{6^2 + 2^2 + 1^2} \\ &= \sqrt{41} \\ |\overrightarrow{\mathrm{BC}}| &= \sqrt{(-1)^2 + (-2)^2 + 3^2} \\ &= \sqrt{14} \\ |\overrightarrow{\mathrm{AC}}| &= \sqrt{5^2 + 0^2 + 4^2} \\ &= \sqrt{41} \end{aligned}$$

AB = AC so $\triangle$ABC is isosceles. $\square$

(c)
$$\begin{aligned} \overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{AC}} &= 25 + 0 + 16 \\ &= 41 \end{aligned}$$

Alternatively,
$$\begin{aligned} \overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{AC}} &= |\overrightarrow{\mathrm{AC}}|^2 \\ &= \left(\sqrt{41}\right)^2 \\ &= 41 \end{aligned}$$

(d)
$$\begin{aligned}\cos\angle\mathrm{A} &= \frac{\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{AC}}}{|\overrightarrow{\mathrm{AB}}||\overrightarrow{\mathrm{AC}}|}\\ &= \frac{30+0+4}{\sqrt{41}\sqrt{41}}\\ &= \frac{34}{41}\\ \angle\mathrm{A} &= 34.0^\circ\\ \angle\mathrm{A}+\angle\mathrm{B}+\angle\mathrm{C} &= 180^\circ\\ \angle\mathrm{C} &= \angle\mathrm{B}\ \text{(isosceles)}\\ \therefore\quad \angle\mathrm{A}+2\angle\mathrm{B} &= 180^\circ\\ 2\angle\mathrm{B} &= 180^\circ - 34.0^\circ\\ &= 146.0^\circ\\ \therefore\quad \angle\mathrm{B}=\angle\mathrm{C} &= 73^\circ\end{aligned}$$

28. Let $\mathbf{v}_\mathrm{A}$ be the velocity of the first bird, and $\mathbf{v}_\mathrm{B}$ that of the second. The apparent velocity of the second from the point of view of the first is

$$\begin{aligned}{}_\mathrm{B}\mathbf{v}_\mathrm{A} &= \mathbf{v}_\mathrm{B}-\mathbf{v}_\mathrm{A}\\ -\mathbf{i}+3\mathbf{j}-\mathbf{k} &= \mathbf{v}_\mathrm{B}-(4\mathbf{i}+\mathbf{j}+\mathbf{k})\\ \mathbf{v}_\mathrm{B} &= (-\mathbf{i}+3\mathbf{j}-\mathbf{k})+(4\mathbf{i}+\mathbf{j}+\mathbf{k})\\ &= (3\mathbf{i}+4\mathbf{j})\,\mathrm{m/s}\\ |\mathbf{v}_\mathrm{B}| &= \sqrt{3^2+4^2+0^2}\\ &= 5\ \mathrm{m/s}\end{aligned}$$

29. If the aircraft are following the same path, their velocity vectors must be scalar multiples. B is behind A but is closing the gap at 35 m/s, so it must be flying 35 m/s faster than A.

$$\begin{aligned}|\mathbf{v}_\mathrm{A}| &= \sqrt{60^2+(-120^2)+40^2}\\ &= 140\ \mathrm{m/s}\\ |\mathbf{v}_\mathrm{B}| &= 140+35\\ &= 175\ \mathrm{m/s}\\ \mathbf{v}_\mathrm{B} &= \frac{175}{140}\mathbf{v}_\mathrm{A}\\ &= \frac{5}{4}\mathbf{v}_\mathrm{A}\\ &= \frac{5}{4}(60\mathbf{i}-120\mathbf{j}+40\mathbf{k})\\ &= (75\mathbf{i}-150\mathbf{j}+50\mathbf{k})\ \ \mathrm{m/s}\end{aligned}$$

Exercise 3B

1. (a) $\mathbf{r} = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{j}+2\mathbf{k})$

(b) $$\begin{cases} x = 3+2\lambda\\ y = 2-\lambda\\ z=-1+2\lambda\end{cases}$$

2. Note that there are many possible correct answers to questions like these. This answer is different to Sadler's. Convince yourself that they are both correct. (What are some other possible correct answers?)

(a) $$\begin{aligned}\overrightarrow{\mathrm{AB}} &= (3\mathbf{i}+\mathbf{j}+\mathbf{k})-(4\mathbf{i}+2\mathbf{j}+3\mathbf{k})\\ &= -\mathbf{i}-\mathbf{j}-2\mathbf{k}\\ \mathbf{r} &= 4\mathbf{i}+2\mathbf{j}+3\mathbf{k}+\lambda\overrightarrow{\mathrm{AB}}\\ &= 4\mathbf{i}+2\mathbf{j}+3\mathbf{k}+\lambda(-\mathbf{i}-\mathbf{j}-2\mathbf{k})\end{aligned}$$

(b) $$\begin{cases} x = 4-\lambda\\ y = 2-\lambda\\ z=3-2\lambda\end{cases}$$

3. $$\begin{aligned}\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+5\mathbf{k}) &= (2\mathbf{i}-3\mathbf{j}+2\mathbf{k})\cdot(3\mathbf{i}-\mathbf{j}+5\mathbf{k})\\ \mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+5\mathbf{k}) &= 6+3+10\\ \mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+5\mathbf{k}) &= 19\end{aligned}$$

4. $$\begin{aligned}\mathbf{r}\cdot\begin{pmatrix}5\\1\\3\end{pmatrix} &= \begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}5\\1\\3\end{pmatrix}\\ \mathbf{r}\cdot\begin{pmatrix}5\\1\\3\end{pmatrix} &= 10+1-9\\ \mathbf{r}\cdot\begin{pmatrix}5\\1\\3\end{pmatrix} &= 2\end{aligned}$$

5. No working required. See Sadler for solution.

6. No working required. See Sadler for solution.

7. Substitute the given point as $\mathbf{r}$:

$$a\mathbf{i}+7\mathbf{j}+5\mathbf{k} = 2\mathbf{i}+b\mathbf{j}-\mathbf{k}+\lambda(-3\mathbf{i}+\mathbf{j}+2\mathbf{k})$$

$\mathbf{k}$ components:

$$\begin{aligned}5 &= -1+2\lambda\\ \lambda &= 3\end{aligned}$$

$\mathbf{i}$ components:

$$\begin{aligned}a &= 2-3\lambda\\ &= 2-3\times 3\\ &= -7\end{aligned}$$

j components:

$$\begin{aligned}7 &= b+\lambda\\ 7 &= b+3\\ b &= 4\end{aligned}$$

8. Substitute $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-1\end{pmatrix} = 21$$

$$3x+2y-z = 21$$

(You should be able to do this by observation in a single step.)

9. No working required. (Use the inverse of the process used for the previous question.)

10. The line is parallel to $-6\mathbf{i}-4\mathbf{j}+2\mathbf{k}$.
$-6\mathbf{i}-4\mathbf{j}+2\mathbf{k} = -2\,(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$
$\therefore$ the line is parallel to $(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$
$(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$ is perpendicular to the plane
$\therefore$ the line is perpendicular to the plane. $\square$

11. Equate the two expressions for **r** and simplify:

$$\begin{pmatrix}10\\5\\-2\end{pmatrix}+\lambda\begin{pmatrix}4\\1\\-2\end{pmatrix} = \begin{pmatrix}0\\8\\-6\end{pmatrix}+\mu\begin{pmatrix}-1\\-3\\5\end{pmatrix}$$

$$\lambda\begin{pmatrix}4\\1\\-2\end{pmatrix}-\mu\begin{pmatrix}-1\\-3\\5\end{pmatrix} = \begin{pmatrix}-10\\3\\-4\end{pmatrix}$$

Equate corresponding components to obtain three equations:

$$\begin{aligned}4\lambda+\mu &= -10\\ \lambda+3\mu &= 3\\ -2\lambda-5\mu &= -4\end{aligned}$$

Solve the first two of these simultaneously to obtain $\lambda=-3, \mu=2$.

Substitute these values into the third equation. If it is consistent, then the two lines intersect.

$-2(-3)-5(2) = -4$ is consistent, so the lines intersect.

Determine the point of intersection by substituting either λ or μ into its original equation:

$$\begin{aligned}\mathbf{r} &= \begin{pmatrix}10\\5\\-2\end{pmatrix}-3\begin{pmatrix}4\\1\\-2\end{pmatrix}\\ &= \begin{pmatrix}-2\\2\\4\end{pmatrix}\end{aligned}$$

12. $$\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}3\\13\\-15\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\4\end{pmatrix}$$

$$\lambda\begin{pmatrix}-1\\3\\2\end{pmatrix}-\mu\begin{pmatrix}-1\\0\\4\end{pmatrix} = \begin{pmatrix}2\\15\\-18\end{pmatrix}$$

Equating corresponding components:

$$\begin{aligned}-\lambda+\mu &= 2\\ 3\lambda &= 15\\ 2\lambda-4\mu &= -18\end{aligned}$$

Solve the first two of these simultaneously to obtain $\lambda=5, \mu=7$.

$2(5)-4(7) = -18$ is consistent, so the lines intersect.

The position vector of the point of intersection is:

$$\begin{aligned}\mathbf{r} &= \mathbf{i}-2\mathbf{j}+3\mathbf{k}+5\,(-\mathbf{i}+3\mathbf{j}+2\mathbf{k})\\ &= -4\mathbf{i}+13\mathbf{j}+13\mathbf{k}\end{aligned}$$

13. (a) $$\begin{pmatrix}13\\1\\8\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}12\\2\\6\end{pmatrix}+\mu\begin{pmatrix}5\\3\\-8\end{pmatrix}$$

$$\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}-\mu\begin{pmatrix}5\\3\\-8\end{pmatrix} = \begin{pmatrix}-1\\1\\-2\end{pmatrix}$$

Equating corresponding components:

$$\begin{aligned}2\lambda-5\mu &= -1\\ -\lambda-3\mu &= 1\\ 3\lambda+8\mu &= -2\end{aligned}$$

Solve the first two of these simultaneously to obtain $\lambda=-\frac{8}{11}, \mu=-\frac{1}{11}$.
$3\left(-\frac{8}{11}\right)+8\left(-\frac{1}{11}\right) = -\frac{32}{11} \neq -2$ is inconsistent, so the lines do not intersect.

(b) $$\begin{pmatrix}13\\1\\8\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}-5\\2\\-3\end{pmatrix}+\beta\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}-\beta\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}-18\\1\\-11\end{pmatrix}$$

Equating corresponding components:

$$\begin{aligned}2\lambda-2\beta &= -18\\ -\lambda-\beta &= 1\\ 3\lambda+\beta &= -11\end{aligned}$$

Solve the first two of these simultaneously to obtain $\lambda=-5, \beta=4$.

$3(-5)+(4) = -11$ is consistent, so the lines intersect.

The position vector of the point of intersection is:

$$\begin{aligned}\mathbf{r} &= 13\mathbf{i}+\mathbf{j}+8\mathbf{k}-5\,(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\\ &= 3\mathbf{i}+6\mathbf{j}-7\mathbf{k}\end{aligned}$$

The angle between the lines is given by

$$\begin{aligned}\cos\theta &= \frac{(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-\mathbf{k})}{|\,(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\,|\,|\,(2\mathbf{i}+\mathbf{j}-\mathbf{k})\,|}\\ &= \frac{4-1-3}{|\,(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\,|\,|\,(2\mathbf{i}+\mathbf{j}-\mathbf{k})\,|}\\ &= 0\end{aligned}$$

The angle between the lines is $90°$.

14. (a) Choose one component (say, $\mathbf{i}$) and solve for λ, then confirm that the same value of λ satisfies both of the other components.

$$\begin{aligned} \mathbf{i}\text{ components:}\quad & -4 = 1 + 5\lambda \\ & \lambda = -1 \\ \mathbf{j}\text{ components:}\quad & -5 = -2 + 3(-1) \quad ✔ \\ \mathbf{k}\text{ components:}\quad & 7 = 5 - 2(-1) \quad ✔ \end{aligned}$$

The same value of λ satisfies all three components, so point A is on line L.

(b) $\mathbf{i}$ components: $10 = 1 + 5\lambda$

$$\lambda = \frac{9}{5}$$

$\mathbf{j}$ components: $3 \neq -2 + 3(\frac{9}{5})$ ✗

The same value of λ does not satisfy both $\mathbf{i}$ and $\mathbf{j}$ components, so point B is not on line L.

(c) Point A:

$$\begin{aligned}(-4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}) \cdot (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) &= 4 - 15 + 14 \\ &= 3 \quad ✔\end{aligned}$$

Point B:

$$\begin{aligned}(10\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) &= -10 + 9 + 4 \\ &= 3 \quad ✔\end{aligned}$$

(d) If line L lies on plane Π then every point on L must satisfy the defining equation for Π.

$$\left(\begin{pmatrix}1\\-2\\5\end{pmatrix} + \lambda\begin{pmatrix}5\\3\\-2\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix} = 3$$

$$\begin{pmatrix}1\\-2\\5\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}5\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix} = 3$$

$$\begin{aligned}-1 - 6 + 10 + \lambda(-5 + 9 - 4) &= 3 \\ 3 + \lambda(0) &= 3\end{aligned}$$

which is true for all λ.

15. $$\overrightarrow{\mathrm{AB}} = t_{\mathrm{A}}\mathbf{v}_{\mathrm{B}}$$

$$\begin{pmatrix}-3\\-8\\2\end{pmatrix} - \begin{pmatrix}-10\\20\\-12\end{pmatrix} = t\left(\begin{pmatrix}5\\-10\\6\end{pmatrix} - \begin{pmatrix}4\\-6\\4\end{pmatrix}\right)$$

$$\begin{pmatrix}7\\-28\\14\end{pmatrix} = t\begin{pmatrix}1\\-4\\2\end{pmatrix}$$

$$t = 7\text{ s}$$

$$\begin{aligned}\mathbf{r} &= \mathbf{r}_{\mathrm{A}} + t\mathbf{v}_{\mathrm{A}} \\ &= \begin{pmatrix}-10\\20\\-12\end{pmatrix} + 7\begin{pmatrix}5\\-10\\6\end{pmatrix} \\ &= \begin{pmatrix}25\\-50\\30\end{pmatrix}\text{ m}\end{aligned}$$

16. $$\begin{aligned}\mathbf{c} &= \begin{pmatrix}2\\2\\1\end{pmatrix} - \begin{pmatrix}1\\2\\0\end{pmatrix} \\ &= \begin{pmatrix}1\\0\\1\end{pmatrix}\end{aligned}$$

Check that $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is not parallel to $\begin{pmatrix}1\\-3\\-5\end{pmatrix}$ ✔

$$\mathbf{r} = \begin{pmatrix}2\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-3\\-5\end{pmatrix} + \mu\begin{pmatrix}1\\0\\1\end{pmatrix}$$

Parametric equations:

$$\begin{cases} x = 2 + \lambda + \mu & ① \\ y = 2 - 3\lambda & ② \\ z = 1 - 5\lambda + \mu & ③ \end{cases}$$

Eliminate λ from ① and ③:

$$\begin{cases} 3x + y = 8 + 3\mu & (3\times①+②)\to④ \\ 5y - 3z = 7 - 3\mu & (5\times②-3\times③)\to⑤ \end{cases}$$

Now eliminate μ and simplify:

$$\begin{aligned} 3x + 6y - 3z &= 15 \qquad (④+⑤) \\ x + 2y - z &= 5 \end{aligned}$$

In scalar product form:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix} = 5$$

$$\mathbf{r}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix} = 5$$

17. $$\begin{aligned}(2\mathbf{i} + 13\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}))\cdot(2\mathbf{i} - \mathbf{j} - \mathbf{k}) &= 11 \\ 4 - 13 - 1 + \lambda(-2 - 3 + 2) &= 11 \\ -10 - 3\lambda &= 11 \\ \lambda &= -7\end{aligned}$$

$$\begin{aligned}\mathbf{r} &= 2\mathbf{i} + 13\mathbf{j} + \mathbf{k} - 7(-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \\ &= 9\mathbf{i} - 8\mathbf{j} + 15\mathbf{k}\end{aligned}$$

18. Let D represent the position of the debris and S the position of the spacecraft, then for a collision to occur

$$\overrightarrow{\mathrm{DS}} = t_{\text{debris}}\mathbf{v}_{\text{spacecraft}}$$

$$\begin{pmatrix}5750\\-13250\\3370\end{pmatrix} - \begin{pmatrix}1200\\3000\\900\end{pmatrix} = t\left(\begin{pmatrix}2000\\-3600\\1000\end{pmatrix} - \begin{pmatrix}600\\1400\\240\end{pmatrix}\right)$$

$$\begin{pmatrix}4550\\-16250\\2470\end{pmatrix} = t\begin{pmatrix}1400\\-5000\\760\end{pmatrix}$$

$$\mathbf{i}: \quad t = \frac{4550}{1400} = 3.25$$

$$\mathbf{j}: \quad t = \frac{-16250}{-5000} = 3.25$$

$$\mathbf{k}: \quad t = \frac{2470}{760} = 3.25$$

The spacecraft and debris will collide at time $t = 3.25$ hours.

19. Position of the fighter at the time of interception is

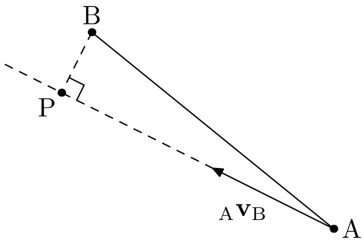
$$\begin{aligned}\mathbf{r} &= (150\mathbf{i} + 470\mathbf{j} + 2\mathbf{k}) + \frac{10}{60}(300\mathbf{i} + 180\mathbf{j}) \\ &= (200\mathbf{i} + 500\mathbf{j} + 2\mathbf{k})\,\text{km}\end{aligned}$$

Call this point P, and call the initial position of the fighter point A, then

$$\begin{aligned}\overrightarrow{\text{AP}} &= (200\mathbf{i} + 500\mathbf{j} + 2\mathbf{k}) - (80\mathbf{i} + 400\mathbf{j} + 3\mathbf{k}) \\ &= (120\mathbf{i} + 100\mathbf{j} - \mathbf{k})\,\text{km}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{AP}} &= t\mathbf{v}_\text{A} \\ 120\mathbf{i} + 100\mathbf{j} - \mathbf{k} &= \frac{10}{60}\mathbf{v}_\text{A} \\ \mathbf{v}_\text{A} &= 6\,(120\mathbf{i} + 100\mathbf{j} - \mathbf{k}) \\ &= (720\mathbf{i} + 600\mathbf{j} - 6\mathbf{k})\ \text{km/h}\end{aligned}$$

20. Let P be the point of minimum separation.



$$\begin{aligned}\overrightarrow{\text{AB}} &= \mathbf{r}_\text{B} - \mathbf{r}_\text{A} \\ &= \begin{pmatrix}2\\40\\26\end{pmatrix} - \begin{pmatrix}30\\-37\\-30\end{pmatrix} \\ &= \begin{pmatrix}-28\\77\\56\end{pmatrix}\end{aligned}$$

$$\begin{aligned}{}_\text{A}\mathbf{v}_\text{B} &= \mathbf{v}_\text{A} - \mathbf{v}_\text{B} \\ &= \begin{pmatrix}5\\8\\3\end{pmatrix} - \begin{pmatrix}8\\0\\-2\end{pmatrix} \\ &= \begin{pmatrix}-3\\8\\5\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{BP}} &= -\overrightarrow{\text{AB}} + t\,{}_\text{A}\mathbf{v}_\text{B} \\ &= -\begin{pmatrix}-28\\77\\56\end{pmatrix} + t\begin{pmatrix}-3\\8\\5\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{BP}} \cdot {}_\text{A}\mathbf{v}_\text{B} &= 0 \\ \left(-\begin{pmatrix}-28\\77\\56\end{pmatrix} + t\begin{pmatrix}-3\\8\\5\end{pmatrix}\right)\cdot\begin{pmatrix}-3\\8\\5\end{pmatrix} &= 0 \\ (-84 - 616 - 280) + t(9 + 64 + 25) &= 0 \\ -980 + 98t &= 0 \\ t &= 10\text{s}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{BP}} &= -\begin{pmatrix}-28\\77\\56\end{pmatrix} + 10\begin{pmatrix}-3\\8\\5\end{pmatrix} \\ &= \begin{pmatrix}-2\\-3\\-6\end{pmatrix} \\ |\overrightarrow{\text{BP}}| &= \sqrt{(-2)^2 + (-3)^2 + (-6)^2} \\ &= 7\text{m}\end{aligned}$$

21. $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is normal to plane Π_1.

$\begin{pmatrix}-2\\2\\-1\end{pmatrix}$ is normal to plane Π_2.

$\begin{pmatrix}-2\\2\\-1\end{pmatrix} = -1\begin{pmatrix}2\\-2\\1\end{pmatrix}$.

$\therefore\quad \begin{pmatrix}-2\\2\\-1\end{pmatrix} \parallel \begin{pmatrix}2\\-2\\1\end{pmatrix}$

$\therefore\quad \begin{pmatrix}2\\-2\\1\end{pmatrix}$ is normal to plane Π_2.

$\therefore\quad \Pi_2 \parallel \Pi_1$ $\square$

Another approach: proof by contradiction.

Suppose Π_1 and Π_2 are not parallel. If that is the case, there exists a line of intersection between the planes, i.e. a set of points $\mathbf{r}$ that simultaneously satisfies

$$\mathbf{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} = 12 \qquad (①)$$

and

$$\mathbf{r}\cdot\begin{pmatrix}-2\\2\\-1\end{pmatrix} = 15 \qquad (②)$$

Starting with ②:

$$\begin{aligned}\mathbf{r}\cdot\begin{pmatrix}-2\\2\\-1\end{pmatrix} &= 15 \\ \mathbf{r}\cdot\left(-1\begin{pmatrix}2\\-2\\1\end{pmatrix}\right) &= 15 \\ -\mathbf{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} &= 15 \\ \mathbf{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} &= -15\end{aligned}$$

substituting ①

$$12 = -15$$

which means that our original supposition leads to a contradiction, hence Π_1 and Π_2 are parallel. □

To find the distance the planes are apart, consider the line

$$\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

We know that this line is perpendicular to both planes, so the distance along this line between the point where it intercepts Π_1 and where it intercepts Π_2 will represent the perpendicular (and hence minimum) distance between the planes. Call these points P_1 and P_2.

$$\begin{aligned}
\lambda_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} &= 12 \\
\lambda_1(4+4+1) &= 12 \\
\lambda_1 &= \frac{4}{3} \\
\mathrm{P}_1 &= \frac{4}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\lambda_2 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} &= 15 \\
\lambda_2(-4-4-1) &= 15 \\
\lambda_2 &= -\frac{5}{3} \\
\mathrm{P}_2 &= -\frac{5}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\mathrm{P}_2\mathrm{P}_1} &= \frac{4}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - -\frac{5}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \\
&= 3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \\
|\overrightarrow{\mathrm{P}_2\mathrm{P}_1}| &= 3\sqrt{2^2 + (-2)^2 + 1^2} \\
&= 9
\end{aligned}$$

The planes are 8 units apart.

22. (a) $\mathbf{r} = (-10000\mathbf{i} - 5000\mathbf{j} + 500\mathbf{k})$
$+ 60\,(80\mathbf{i} + 50\mathbf{j} + 5\mathbf{k})$
$= (-5200\mathbf{i} - 2000\mathbf{j} + 800\mathbf{k})$

(b) When it is due west, the $\mathbf{j}$ component will be zero.
$-5\,000 + 50t = 0$
$t = 100\text{s}$
i.e. at 1 minute and 40 seconds after 1pm.

(c) When it is due north, the $\mathbf{i}$ component will be zero.
$-10\,000 + 80t = 0$
$t = 125\text{s}$
i.e. at 2 minutes and 5 seconds after 1pm.

The altitude at that time is given by the $\mathbf{k}$ component:
$\mathbf{500} + 5 \times 125t = 1\,125\text{m}$

(d) Five minutes is 300 seconds:

$$\begin{aligned}
\mathbf{r} &= (-10000\mathbf{i} - 5000\mathbf{j} + 500\mathbf{k}) \\
&\quad + 300\,(80\mathbf{i} + 50\mathbf{j} + 5\mathbf{k}) \\
&= (14000\mathbf{i} + 10000\mathbf{j} + 2000\mathbf{k}) \\
\text{distance} &= \sqrt{14000^2 + 10000^2 + 2000^2} \\
&= 2000\sqrt{7^2 + 5^2 + 1^2} \\
&= 10000\sqrt{3}\text{m} \\
&= 10\sqrt{3}\text{km}
\end{aligned}$$

(e) Let P be the position nearest O.

$$\overrightarrow{\mathrm{OP}} = \begin{pmatrix} -10000 \\ -5000 \\ 500 \end{pmatrix} + t \begin{pmatrix} 80 \\ 50 \\ 5 \end{pmatrix}$$

$$\begin{aligned}
\overrightarrow{\mathrm{OP}} \cdot \begin{pmatrix} 80 \\ 50 \\ 5 \end{pmatrix} &= 0 \\
\left(\begin{pmatrix} -10000 \\ -5000 \\ 500 \end{pmatrix} + t \begin{pmatrix} 80 \\ 50 \\ 5 \end{pmatrix} \right) \cdot \begin{pmatrix} 80 \\ 50 \\ 5 \end{pmatrix} &= 0 \\
(-800000 - 250000 + 2500) & \\
+t(6400 + 2500 + 25) &= 0 \\
-1052500 + 8925t &= 0 \\
t &= 118\text{s}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\mathrm{OP}} &= \begin{pmatrix} -10000 \\ -5000 \\ 500 \end{pmatrix} + 118 \begin{pmatrix} 80 \\ 50 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} -566 \\ 896 \\ 1090 \end{pmatrix} \\
|\overrightarrow{\mathrm{OP}}| &= \sqrt{(-566)^2 + (896)^2 + (1090)^2} \\
&= 1520\text{m} \\
&= 1.52\text{km}
\end{aligned}$$

Miscellaneous Exercise 3

1. (a) No working needed. Refer to the answer in Sadler.

   (b) No working needed. Refer to the answer in Sadler.

   (c)
$$\begin{aligned}\cos\theta &= \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}\\ &= \frac{2\times -2+3\times 4+-2\times 3}{\sqrt{2^2+3^2+(-2)^2}\sqrt{(-2)^2+4^2+3^2}}\\ &= \frac{2}{\sqrt{17}\sqrt{29}}\\ \theta &= 85^\circ\end{aligned}$$

   (d)
$$\begin{aligned}\cos\theta &= \frac{\mathbf{p}\cdot\mathbf{i}}{|\mathbf{p}||\mathbf{i}|}\\ &= \frac{2}{\sqrt{17}}\\ \theta &= 61^\circ\end{aligned}$$

   (e)
$$\begin{aligned}\cos\theta &= \frac{\mathbf{q}\cdot\mathbf{j}}{|\mathbf{q}||\mathbf{j}|}\\ &= \frac{4}{\sqrt{29}}\\ \theta &= 42^\circ\end{aligned}$$

2.
$$\begin{pmatrix}2\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-2\end{pmatrix}=\begin{pmatrix}-2\\1\\6\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\2\end{pmatrix}$$
$$\lambda\begin{pmatrix}2\\3\\-2\end{pmatrix}-\mu\begin{pmatrix}-2\\-1\\2\end{pmatrix}=\begin{pmatrix}-4\\1\\4\end{pmatrix}$$
$$\begin{aligned}2\lambda+2\mu &= -4 && ①\\ 3\lambda+\mu &= 1 && ②\\ -2\lambda-2\mu &= 4 && ③\\ 4\lambda &= 6 && 2\times ② + ③\\ \lambda &= \frac{3}{2}\\ -2\left(\frac{3}{2}\right)-2\mu &= 4 && \text{subst. } ③\\ \mu &= -\frac{7}{2}\end{aligned}$$

   We would normally need to confirm that these values work in the third equation (i.e. ①) but in this case ① and ③ are redundant (i.e. they can be rearranged to be identical equations) so any solution of ③ must also be a solution of ①. (This also means that we didn't really need to find μ.)

   The point of intersection is given by:
$$\begin{aligned}\mathbf{r} &= 2\mathbf{i}+2\mathbf{k}+\lambda(2\mathbf{i}+3\mathbf{j}-2\mathbf{k})\\ &= 2\mathbf{i}+2\mathbf{k}+\frac{3}{2}(2\mathbf{i}+3\mathbf{j}-2\mathbf{k})\\ &= 5\mathbf{i}+4.5\mathbf{j}-\mathbf{k}\end{aligned}$$

3. The resultant is
$$(\mathbf{i}+5\mathbf{j}-4\mathbf{k})+(\mathbf{i}-3\mathbf{j}+3\mathbf{k})=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$$

   This has magnitude
$$\sqrt{2^2+2^2+(-1)^2}=3$$
   so a unit vector parallel to the resultant is
$$\frac{1}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$$

4. $\mathrm{f}(x)=x^2+3x$
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{h\to 0}\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}\\ &= \lim_{h\to 0}\frac{\big((x+h)^2+3(x+h)\big)-\big(x^2+3x\big)}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2+3x+3h-x^2-3x}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2+3h}{h}\\ &= \lim_{h\to 0}(2x+h+3)\\ &= 2x+3\end{aligned}$$

5. (a)
$$\begin{aligned}z &= \frac{3+5\sqrt{3}\mathrm{i}}{-3+2\sqrt{3}\mathrm{i}}\\ &= \frac{3+5\sqrt{3}\mathrm{i}}{-3+2\sqrt{3}\mathrm{i}}\,\frac{-3-2\sqrt{3}\mathrm{i}}{-3-2\sqrt{3}\mathrm{i}}\\ &= \frac{-9-6\sqrt{3}\mathrm{i}-15\sqrt{3}\mathrm{i}+30}{9+6\sqrt{3}\mathrm{i}-6\sqrt{3}\mathrm{i}+12}\\ &= \frac{21-21\sqrt{3}\mathrm{i}}{21}\\ &= 1-\sqrt{3}\mathrm{i}\end{aligned}$$

   (b)
$$\begin{aligned}r &= \sqrt{1^2+(-\sqrt{3})^2}\\ &= 2\end{aligned}$$
   θ is in the 4th quadrant (positive real component, negative imaginary component) and
$$\begin{aligned}\tan\theta &= \frac{-\sqrt{3}}{1}\\ \theta &= -\frac{\pi}{3}\\ \therefore\quad z &= 2\operatorname{cis}\left(-\frac{\pi}{3}\right)\end{aligned}$$

6.
$$\begin{aligned}\cos\theta &= \frac{\begin{pmatrix}2\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\1\end{pmatrix}}{\left|\begin{pmatrix}2\\3\\-2\end{pmatrix}\right|\left|\begin{pmatrix}3\\-2\\1\end{pmatrix}\right|}\\ &= \frac{-2}{\sqrt{17}\sqrt{14}}\\ \theta &= 97.4^\circ\end{aligned}$$

7. $$\begin{aligned} 4\operatorname{cis}\frac{-\pi}{6} &= 4\cos\frac{-\pi}{6} + \left(4\sin\frac{-\pi}{6}\right)\mathrm{i} \\ &= 2\sqrt{3} - 2\mathrm{i} \\ \frac{1}{4\operatorname{cis}\frac{-\pi}{6}} &= \frac{1}{2\sqrt{3}-2\mathrm{i}} \\ &= \frac{1}{2\sqrt{3}-2\mathrm{i}}\,\frac{2\sqrt{3}+2\mathrm{i}}{2\sqrt{3}+2\mathrm{i}} \\ &= \frac{2\sqrt{3}+2\mathrm{i}}{12+4\sqrt{3}\mathrm{i}-4\sqrt{3}\mathrm{i}+4} \\ &= \frac{2\sqrt{3}+2\mathrm{i}}{16} \\ &= \frac{\sqrt{3}}{8} + \frac{1}{8}\mathrm{i} \end{aligned}$$

Alternatively:

$$\begin{aligned} \frac{1}{4\operatorname{cis}\frac{-\pi}{6}} &= \frac{1}{4}\operatorname{cis}\left(0 - \frac{-\pi}{6}\right) \\ &= \frac{1}{4}\operatorname{cis}\frac{\pi}{6} \\ &= \frac{1}{4}\left(\cos\frac{\pi}{6}\right) + \frac{1}{4}\left(\sin\frac{\pi}{6}\right)\mathrm{i} \\ &= \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right)\mathrm{i} \\ &= \frac{\sqrt{3}}{8} + \frac{1}{8}\mathrm{i} \end{aligned}$$

8. No working required. If you're having trouble understanding this question, think about what *operation* a 90° rotation in the Argand plane represents.

9. (a) The first line is parallel to $(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ and the second parallel to $(6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$. Since these are scalar multiples, the lines are parallel.

   (b) First observe that the lines are not parallel. Next equate the lines and rearrange to make three equations from the three components:

   $$\begin{aligned} -\lambda - 2\mu &= -7 \\ 3\lambda - \mu &= 0 \\ \lambda - 2\mu &= -15 \end{aligned}$$

   Solve the first two simultaneously

   $$\begin{aligned} \lambda &= 1 \\ \mu &= 3 \end{aligned}$$

   Does this solution satisfy the third equation?

   $$(1) - 2(3) \neq -15$$

   No: they are skew lines.

   (c) First observe that the lines are not parallel. Next equate the lines and rearrange to make three equations from the three components:

   $$\begin{aligned} -\lambda - \mu &= 1 \\ \lambda &= -3 \\ \lambda - 2\mu &= -7 \end{aligned}$$

   Solve the first two simultaneously

   $$\begin{aligned} \lambda &= -3 \\ \mu &= 2 \end{aligned}$$

   Does this solution satisfy the third equation?

   $$(-3) - 2(2) = -7$$

   Yes: the lines intersect.

   (d) The first line is parallel to $(\mathbf{i} + \mathbf{j} - \mathbf{k})$ and the second parallel to $(-\mathbf{i} - \mathbf{j} + \mathbf{k})$. Since these are scalar multiples, the lines are parallel.

10. The height corresponds to the $\mathbf{k}$ component, so

$$\begin{aligned} 3\lambda &= 180 \\ \lambda &= 60 \end{aligned}$$

and the initial position is

$$\begin{aligned} \mathbf{r} &= 60\,(10\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) \\ &= (600\mathbf{i} + 240\mathbf{j} + 180\mathbf{k})\,\mathrm{m} \end{aligned}$$

The displacement from there to the touchdown point at $(0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k})\,\mathrm{m}$ is

$$\begin{aligned} \mathbf{s} &= (\mathbf{0}) - (600\mathbf{i} + 240\mathbf{j} + 180\mathbf{k}) \\ &= (-600\mathbf{i} - 240\mathbf{j} - 180\mathbf{k})\,\mathrm{m} \end{aligned}$$

giving a velocity of

$$\begin{aligned} \mathbf{v} &= \frac{1}{15}(-600\mathbf{i} - 240\mathbf{j} - 180\mathbf{k}) \\ &= (-40\mathbf{i} - 16\mathbf{j} - 12\mathbf{k})\,\mathrm{m/s} \end{aligned}$$

The distance travelled is

$$\begin{aligned} d &= |\,(-600\mathbf{i} - 240\mathbf{j} - 180\mathbf{k})\,| \\ &= 60\sqrt{10^2 + 4^2 + 3^2} \\ &= 670\mathrm{m} \end{aligned}$$

11. (a) $$\begin{aligned} zw &= (2)(1)\operatorname{cis}\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \\ &= 2\operatorname{cis}\frac{5\pi}{12} \end{aligned}$$

   (b) $$\begin{aligned} \frac{z}{w} &= \frac{2}{1}\operatorname{cis}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= 2\operatorname{cis}\frac{\pi}{12} \end{aligned}$$

   (c) $$\begin{aligned} w^2 &= 1^2\operatorname{cis}\left(2 \times \frac{\pi}{6}\right) \\ &= \operatorname{cis}\frac{\pi}{3} \end{aligned}$$

(d)
$$\begin{aligned} z^3 &= 2^3 \operatorname{cis}\left(3 \times \frac{\pi}{4}\right) \\ &= 8 \operatorname{cis} \frac{3\pi}{4} \end{aligned}$$

(e)
$$\begin{aligned} w^9 &= 1^9 \operatorname{cis}\left(9 \times \frac{\pi}{6}\right) \\ &= \operatorname{cis} \frac{3\pi}{2} \\ &= \operatorname{cis}\left(-\frac{\pi}{2}\right) \end{aligned}$$

(f)
$$\begin{aligned} z^9 &= 2^9 \operatorname{cis}\left(9 \times \frac{\pi}{4}\right) \\ &= 512 \operatorname{cis} \frac{9\pi}{4} \\ &= 512 \operatorname{cis} \frac{\pi}{4} \end{aligned}$$

12. The first part of this question is really a 2D question, since "passes directly over" means we are not concerned with depth, so ignore the $\mathbf{k}$ component until it comes to finding the depth of the submarine.

$$\begin{aligned} (1150\mathbf{i} + 827\mathbf{j}) + t(10\mathbf{i} - 2\mathbf{j}) &= (1345\mathbf{i} + 970\mathbf{j}) + t(-5\mathbf{i} - 13\mathbf{j}) \\ t((10\mathbf{i} - 2\mathbf{j}) - (-5\mathbf{i} - 13\mathbf{j})) &= (1345\mathbf{i} + 970\mathbf{j}) - (1150\mathbf{i} + 827\mathbf{j}) \\ t(15\mathbf{i} + 11\mathbf{j}) &= (195\mathbf{i} + 143\mathbf{j}) \\ t &= 13 \end{aligned}$$

satisfies both components, so the tanker passes over the submarine at $t = 13$ seconds. The depth of the submarine at that time is

$$\begin{aligned} d &= 4 \times 13 \\ &= 52 \text{ m below the surface.} \end{aligned}$$

13. The plane $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = -14$ is perpendicular to the line $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$.

The plane $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 42$ is perpendicular to the line $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$.

Two planes perpendicular to the same line are be parallel to each other. Therefore the two planes are parallel.

The perpendicular line $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ intersects plane $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = -14$ at point A:

$$\begin{aligned} \left(\lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} &= -14 \\ \lambda(4 + 9 + 36) &= -14 \\ 49\lambda &= -14 \\ \lambda &= -\frac{2}{7} \\ \mathbf{r}_{\mathrm{A}} &= -\frac{2}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \end{aligned}$$

The same line intersects plane $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 42$ at point B:

$$\begin{aligned} \left(\lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} &= 42 \\ \lambda(4 + 9 + 36) &= 42 \\ 49\lambda &= 42 \\ \lambda &= \frac{6}{7} \\ \mathbf{r}_{\mathrm{B}} &= \frac{6}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \overrightarrow{\mathrm{AB}} &= \frac{6}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} - -\frac{2}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \\ &= \frac{8}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \\ |\overrightarrow{\mathrm{AB}}| &= \frac{8}{7}\sqrt{2^2 + (-3)^2 + 6^2} \\ &= 8 \end{aligned}$$

The distance between the planes is 8 units.

We can generalise this result. Suppose we have a pair of parallel planes expressed as

$$\begin{aligned} \mathbf{r} \cdot \mathbf{n} &= a \\ \text{and} \quad \mathbf{r} \cdot \mathbf{n} &= b \end{aligned}$$

These planes are perpendicular to the line

$$\mathbf{r} = \lambda \mathbf{n}$$

The points of intersection between the planes and this line are given by

$$\begin{aligned} (\lambda_{\mathrm{A}} \mathbf{n}) \cdot \mathbf{n} &= a \\ \lambda_{\mathrm{A}} &= \frac{a}{\mathbf{n} \cdot \mathbf{n}} \\ \mathbf{r}_{\mathrm{A}} &= \lambda_{\mathrm{A}} \mathbf{n} \\ &= \frac{a}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \\ &= \frac{a}{|\mathbf{n}|^2} \mathbf{n} \end{aligned}$$

similarly

$$\begin{aligned}
\mathbf{r}_\mathrm{B} &= \frac{b}{|\mathbf{n}|^2}\mathbf{n} \\
\overrightarrow{\mathrm{AB}} &= \mathbf{r}_\mathrm{B} - \mathbf{r}_\mathrm{A} \\
&= \frac{b-a}{|\mathbf{n}|^2}\mathbf{n} \\
|\overrightarrow{\mathrm{AB}}| &= \frac{|b-a|}{|\mathbf{n}|^2}|\mathbf{n}| \\
&= \frac{|b-a|}{|\mathbf{n}|}
\end{aligned}$$

14. Let P be the point on the line nearest the origin. $\overrightarrow{\mathrm{OP}}$ is perpendicular to the line.

$$\begin{aligned}
\left(\begin{pmatrix}2\\3\\-1\end{pmatrix} + \lambda_\mathrm{P}\begin{pmatrix}5\\2\\1\end{pmatrix}\right)\cdot\begin{pmatrix}5\\2\\1\end{pmatrix} &= 0 \\
(10+6-1) + \lambda_\mathrm{P}(25+4+1) &= 0 \\
15 + 30\lambda_\mathrm{P} &= 0 \\
\lambda_\mathrm{P} &= -0.5
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\mathrm{OP}} &= \begin{pmatrix}2\\3\\-1\end{pmatrix} - 0.5\begin{pmatrix}5\\2\\1\end{pmatrix} \\
&= \begin{pmatrix}-0.5\\2\\-1.5\end{pmatrix} \\
|\overrightarrow{\mathrm{OP}}| &= \sqrt{(-0.5)^2 + 2^2 + (-1.5)^2} \\
&= \frac{1}{2}\sqrt{(-1)^2 + 4^2 + (-3)^2} \\
&= \frac{\sqrt{26}}{2}
\end{aligned}$$

15. Let A be the point $\begin{pmatrix}1\\0\\-4\end{pmatrix}$. Let P be the point on the line nearest to A. $\overrightarrow{\mathrm{AP}}$ is perpendicular to the line.

$$\begin{aligned}
\overrightarrow{\mathrm{OP}} &= \begin{pmatrix}-3\\-7\\8\end{pmatrix} + \lambda_\mathrm{P}\begin{pmatrix}3\\4\\-5\end{pmatrix} \\
\overrightarrow{\mathrm{AP}} &= \begin{pmatrix}-3\\-7\\8\end{pmatrix} + \lambda_\mathrm{P}\begin{pmatrix}3\\4\\-5\end{pmatrix} - \begin{pmatrix}1\\0\\-4\end{pmatrix} \\
&= \begin{pmatrix}-4\\-7\\12\end{pmatrix} + \lambda_\mathrm{P}\begin{pmatrix}3\\4\\-5\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\left(\begin{pmatrix}-4\\-7\\12\end{pmatrix} + \lambda_\mathrm{P}\begin{pmatrix}3\\4\\-5\end{pmatrix}\right)\cdot\begin{pmatrix}3\\4\\-5\end{pmatrix} &= 0 \\
(-12-28-60) + \lambda_\mathrm{P}(9+16+25) &= 0 \\
-100 + 50\lambda_\mathrm{P} &= 0 \\
\lambda_\mathrm{P} &= 2
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\mathrm{AP}} &= \begin{pmatrix}-4\\-7\\12\end{pmatrix} + 2\begin{pmatrix}3\\4\\-5\end{pmatrix} \\
&= \begin{pmatrix}2\\1\\2\end{pmatrix} \\
\mathrm{AP} &= \sqrt{2^2 + 1^2 + 2^2} \\
&= 3 \text{ units}
\end{aligned}$$