Chapter 8

Exercise 8A

1-9 No working required. You should be able to differentiate these by observation.

10. $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x\sqrt{x} = \mathrm{e}^x\sqrt{x} + \mathrm{e}^x\left(-\frac{1}{2\sqrt{x}}\right) = \frac{\mathrm{e}^x(2x-1)}{2\sqrt{x}}$$

11. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x\sin x &= \mathrm{e}^x\sin x + \mathrm{e}^x\cos x\\ &= \mathrm{e}^x(\sin x + \cos x)\\ &= \sqrt{2}\mathrm{e}^x\sin\left(x+\frac{\pi}{4}\right)\end{aligned}$$
The last step is not really necessary here. You should, however, understand how it is obtained. (But see question 16.)

12. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x\cos 2x &= \mathrm{e}^x\cos 2x - 2\mathrm{e}^x\sin 2x\\ &= \mathrm{e}^x(\cos 2x - 2\sin 2x)\end{aligned}$$

13. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x\sin^2 x &= \mathrm{e}^x\sin^2 x + \mathrm{e}^x 2\sin x\cos x\\ &= \mathrm{e}^x\sin x(\sin x + 2\cos x)\end{aligned}$$

14. Simply apply the chain rule and you should be able to do this in a single step. No working required.

15. Again, one step using the chain rule.

16. $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{2}\mathrm{e}^x\sin\left(x+\frac{\pi}{4}\right)$ (see question 11). Where the gradient is zero,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \sqrt{2}\mathrm{e}^x\sin\left(x+\frac{\pi}{4}\right) &= 0\\ \sin\left(x+\frac{\pi}{4}\right) &= 0\\ x+\frac{\pi}{4} &= 0 + k\pi \qquad (k\in\mathbb{I})\\ x &= -\frac{\pi}{4}+k\pi\\ x &\in \left\{-\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}\right\}\end{aligned}$$

17. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= (\mathrm{e}^x(\sin x + \cos x))(2)\\ &= 2\mathrm{e}^x(\sin x + \cos x)\end{aligned}$$
When $x = \pi$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= 2\mathrm{e}^\pi(\sin\pi + \cos\pi)\\ &= -2\mathrm{e}^\pi\end{aligned}$$

18. (a)
$$\begin{aligned}\mathrm{e}^x\sin y &= x\\ \mathrm{e}^x\sin y + \mathrm{e}^x\cos y\frac{\mathrm{d}y}{\mathrm{d}x} &= 1\\ \mathrm{e}^x\cos y\frac{\mathrm{d}y}{\mathrm{d}x} &= 1-\mathrm{e}^x\sin y\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1-\mathrm{e}^x\sin y}{\mathrm{e}^x\cos y}\end{aligned}$$

(b)
$$\begin{aligned}y &= x + \mathrm{e}^{x+y}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 + \mathrm{e}^{x+y}\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ &= 1 + \mathrm{e}^{x+y} + \mathrm{e}^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x}\\ \frac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1+\mathrm{e}^{x+y}\\ \frac{\mathrm{d}y}{\mathrm{d}x}\left(1-\mathrm{e}^{x+y}\right) &= 1+\mathrm{e}^{x+y}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1+\mathrm{e}^{x+y}}{1-\mathrm{e}^{x+y}}\end{aligned}$$

(c)
$$\begin{aligned}x^2 + xy &= 2 + y^2\mathrm{e}^x\\ 2x + y + x\frac{\mathrm{d}y}{\mathrm{d}x} &= 2y\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^x + y^2\mathrm{e}^x\\ x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2\mathrm{e}^x - 2x - y\\ \frac{\mathrm{d}y}{\mathrm{d}x}(x-2y\mathrm{e}^x) &= y^2\mathrm{e}^x - 2x - y\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y^2\mathrm{e}^x - 2x - y}{x - 2y\mathrm{e}^x}\end{aligned}$$

Exercise 8B

1. No working needed: integrate by observation.

2. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{6x} = 6\mathrm{e}^{6x}$ so $\int \mathrm{e}^{6x}\,\mathrm{d}x = \frac{\mathrm{e}^{6x}}{6} + c$

3. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{2x} = 2\mathrm{e}^{2x}$ so $\int 5\mathrm{e}^{2x}\,\mathrm{d}x = \frac{5\mathrm{e}^{2x}}{2} + c$

4. $\frac{1}{\mathrm{e}^x} = \mathrm{e}^{-x}$ so
$$\begin{aligned}\int\frac{6}{e^x}\,\mathrm{d}x &= \int 6e^{-x}\,\mathrm{d}x\\ &= -6\mathrm{e}^{-x} + c\\ &= -\frac{6}{e^x} + c\end{aligned}$$

5. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{0.5x} = 0.5\mathrm{e}^{0.5x}$ so $\int 8\mathrm{e}^{0.5x}\,\mathrm{d}x = 16\mathrm{e}^{0.5x} + c$

6. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\mathrm{e}^x} = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{0.5x} = 0.5\mathrm{e}^{0.5x}$ so

$$\int 2\sqrt{\mathrm{e}^x}\,\mathrm{d}x = 4\sqrt{\mathrm{e}^x} + c$$

7. $$\begin{aligned}\int (6\mathrm{e}^{3x} + 2x)dx &= \int 6\mathrm{e}^{3x}\,\mathrm{d}x + \int 2x\,\mathrm{d}x\\ &= 2\mathrm{e}^{3x} + x^2 + c\end{aligned}$$

8. $$\begin{aligned}\int (2\mathrm{e}^{3x} + 3\mathrm{e}^{2x})dx &= \int 2\mathrm{e}^{3x}\,\mathrm{d}x + \int 3\mathrm{e}^{2x}\,\mathrm{d}x\\ &= \frac{2\mathrm{e}^{3x}}{3} + \frac{3\mathrm{e}^{2x}}{2} + c\end{aligned}$$

9. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-2x} = -2\mathrm{e}^{-2x}$ so $\int 4\mathrm{e}^{2x}\,\mathrm{d}x = -2\mathrm{e}^{-2x} + c$

10. $$\begin{aligned}\int \left(\frac{3}{\mathrm{e}^{2x}} + \frac{\mathrm{e}^{2x}}{3}\right) dx &= \int \frac{3}{\mathrm{e}^{2x}}\,\mathrm{d}x + \int \frac{\mathrm{e}^{2x}}{3}\,\mathrm{d}x\\ &= \int 3\mathrm{e}^{-2x}\,\mathrm{d}x + \frac{e^{2x}}{6} + c\\ &= -\frac{3}{2}\mathrm{e}^{-2x} + \frac{e^{2x}}{6} + c\\ &= \frac{\mathrm{e}^{2x}}{6} - \frac{3}{2\mathrm{e}^2x} + c\end{aligned}$$

11. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{3x^2} = 6x\mathrm{e}^{3x^2}$ so

$$\int 12x\mathrm{e}^{3x^2}\,\mathrm{d}x = 2\mathrm{e}^{3x^2} + c$$

12. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{3x-2} = 3\mathrm{e}^{3x-2}$ so

$$\int 6x\mathrm{e}^{3x-2}\,\mathrm{d}x = 2\mathrm{e}^{3x-2} + c$$

With a little practice you should be able to do problems like this by observation. Questions 11–15 are all in the form

$$\int a\mathrm{f}'(x)\mathrm{e}^{f(x)}\,\mathrm{d}x$$

so all that needs to be done is to determine what $f(x)$ and a are then

$$\int a\mathrm{f}'(x)\mathrm{e}^{f(x)}\,\mathrm{d}x = a\mathrm{e}^{f(x)}$$

(essentially the chain rule in reverse). For questions 13–15 these solutions will simply state $f(x)$ since the rest should be obvious.

13. $f(x) = x^2 + 1$

14. $f(x) = \sin x$

15. $f(x) = 2\sin x$

To answer questions 16–20 you should use

$$\int a\mathrm{f}'(x)\,(\mathrm{f}(x))^n = a\frac{(f(x))^{n+1}}{n+1} + c$$

For these questions these solutions will simply state $f(x)$ and $f'(x)$ since the rest should be obvious.

16. $\mathrm{f}(x) = 1 + \mathrm{e}^x \qquad \mathrm{f}'(x) = \mathrm{e}^x$

17. $\mathrm{f}(x) = 1 + 2\mathrm{e}^x \qquad \mathrm{f}'(x) = 2\mathrm{e}^x$

18. $\mathrm{f}(x) = 1 + \mathrm{e}^x \qquad \mathrm{f}'(x) = e^x \qquad (n = \frac{1}{2})$

19. $\mathrm{f}(x) = 1 - 2\mathrm{e}^x \qquad \mathrm{f}'(x) = -2e^x$

20. $\mathrm{f}(x) = 2 + \mathrm{e}^{\sin x} \qquad \mathrm{f}'(x) = \cos x\,\mathrm{e}^{\sin x}$

21–23 No working needed.

24. $$\begin{aligned}\int_0^3 3\mathrm{e}^x\,\mathrm{d}x &= [3\mathrm{e}^x]_0^3\\ &= 3\mathrm{e}^3 - 3\mathrm{e}^0\\ &= 3(\mathrm{e}^3 - 1)\end{aligned}$$

25. $$\begin{aligned}\int_0^2 \left(2\mathrm{e}^x + 3\mathrm{e}^{2x}\right) dx &= \left[2\mathrm{e}^x + \frac{3\mathrm{e}^{2x}}{2}\right]_0^2\\ &= \left(2\mathrm{e}^2 + \frac{3\mathrm{e}^4}{2}\right) - \left(2\mathrm{e}^0 + \frac{3\mathrm{e}^0}{2}\right)\\ &= \left(2\mathrm{e}^2 + \frac{3\mathrm{e}^4}{2}\right) - \left(2 + \frac{3}{2}\right)\\ &= \left(2\mathrm{e}^2 + \frac{3\mathrm{e}^4}{2}\right) - \frac{7}{2}\\ &= \frac{1}{2}(4\mathrm{e}^2 + 3\mathrm{e}^4 - 7)\\ &= \frac{1}{2}(3e^2 + 7)(e^2 - 1)\end{aligned}$$

The factorization in the last step is not strictly necessary, but since the result factors nicely and since factor form is frequently more useful than expanded form it is reasonable to leave the result thus.

26. $$\begin{aligned}\int_0^{\frac{\pi}{3}} 8\sin x\,\mathrm{e}^{4\cos x}\,\mathrm{d}x &= \left[-2\mathrm{e}^{4\cos x}\right]_0^{\frac{\pi}{3}}\\ &= (-2\mathrm{e}^{4\cos\frac{\pi}{3}}) - (-2\mathrm{e}^{4\cos 0})\\ &= -2\mathrm{e}^2 + 2\mathrm{e}^4\\ &= 2e^2(e^2 - 1)\end{aligned}$$

27. (a) $$\begin{aligned}\mathrm{f}(x) &= \int 4x - 6\mathrm{e}^{3x}\,\mathrm{d}x\\ &= 2x^2 - 2\mathrm{e}^{3x} + c\\ \mathrm{f}(0) &= 3\\ -2\mathrm{e}^0 + c &= 3\\ c &= 5\\ \therefore\quad \mathrm{f}(x) &= 2x^2 - 2\mathrm{e}^{3x} + 5\end{aligned}$$

(b) $$\begin{aligned}\mathrm{f}(1) &= 2\times 1^2 - 2\mathrm{e}^{3\times 1} + 5\\ &= 2 - 2\mathrm{e}^3 + 7\\ &= 7 - 2\mathrm{e}^3\end{aligned}$$

Exercise 8C

1-29 No working required. You should be able to do all these in a single step.

30. $$\log_2 21 = \frac{\ln 21}{\ln 2} = \frac{\ln(3\times 7)}{\ln 2} = \frac{\ln 3 + \ln 7}{\ln 2}$$

31. $$\log_3 200 = \frac{\ln 200}{\ln 3} = \frac{\ln(2^3\times 5^2)}{\ln 3} = \frac{3\ln 2 + 2\ln 5}{\ln 3}$$

32. $$\log_5 50 = \log_5(5^2\times 2) = \log_5(5^2) + \log_5 2 = 2 + \frac{\ln 2}{\ln 5}$$

33. $$\log_6 9 = \frac{\ln 3^2}{\ln(3\times 2)} = \frac{2\ln 3}{\ln 3 + \ln 2}$$

34. $$\log_9 6 = \frac{\ln(3\times 2)}{\ln 3^2} = \frac{\ln 3 + \ln 2}{2\ln 3} = \frac{1}{2} + \frac{\ln 2}{2\ln 3}$$

35. $$\log_4 300 = \frac{\ln(2^2\times 3\times 5^2)}{\ln 2^2} = \frac{2\ln 2 + \ln 3 + 2\ln 5}{2\ln 2} = 1 + \frac{\ln 3 + 2\ln 5}{2\ln 2}$$

36. $$\log_8 220 = \frac{\ln(2^2\times 5\times 11)}{\ln 2^3} = \frac{2\ln 2 + \ln 5 + \ln 11}{3\ln 2} = \frac{2}{3} + \frac{\ln 5 + \ln 11}{3\ln 2}$$

37. $$\begin{aligned} e^{x+1} &= 12 \\ x+1 &= \ln 12 \\ x &= \ln 12 - 1 \end{aligned}$$

38. $$\begin{aligned} e^{x+2} &= 25 \\ x+2 &= \ln 25 \\ x &= \ln 25 - 2 \end{aligned}$$

39. $$\begin{aligned} e^{x-1} &= 150 \\ x-1 &= \ln 150 \\ x &= \ln 150 + 1 \end{aligned}$$

40. $$\begin{aligned} e^{2x+1} &= 34 \\ 2x+1 &= \ln 34 \\ x &= \frac{\ln 34 - 1}{2} \end{aligned}$$

41. $$\begin{aligned} 5e^{x+1} + 3e^{x+1} &= 200 \\ 8e^{x+1} &= 200 \\ e^{x+1} &= 25 \\ x+1 &= \ln 25 \\ x &= \ln 25 - 1 \end{aligned}$$

42. $$\begin{aligned} e^{2x} - 12e^{x} &= -35 \\ (e^x)^2 - 12e^x + 35 &= 0 \\ (e^x - 5)(e^x - 7) &= 0 \\ e^x &= 5 \quad \text{or} \quad e^x = 7 \\ x &= \ln 5 \qquad\quad x = \ln 7 \end{aligned}$$

43. $$3\log x + \log y = \log x^3 + \log y = \log(x^3 y)$$

44. $$2\log x - 3\log y = \log x^2 - \log y^3 = \log\frac{x^2}{y^3}$$

45. $$2\log a + \log b - 3\log c = \log a^2 + \log b - \log c^3 = \log\frac{a^2 b}{c^3}$$

46. $$3 + \log x = \log 10^3 + \log x = \log(1000x)$$

47. $$2 + \ln x = \ln e^2 + \ln x = \ln(e^2 x)$$

48. $$3 - \ln x + 2\ln y = \ln e^3 - \ln x + \ln y^2 = \ln\frac{e^3 y^2}{x}$$

Exercise 8D

1. $$\begin{aligned} y &= \ln 5x \\ &= \ln 5 + \ln x \\ \frac{dy}{dx} &= \frac{1}{x} \end{aligned}$$

2. $$\begin{aligned} y &= 3x + \ln 3x \\ &= 3x + \ln 3 + \ln x \\ \frac{dy}{dx} &= 3 + \frac{1}{x} \end{aligned}$$

3. $$\frac{dy}{dx} = \frac{2}{x}$$

4. $$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2x+3}(2) \\ &= \frac{2}{2x+3} \end{aligned}$$

5. $$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2x-3}(2) \\ &= \frac{2}{2x-3} \end{aligned}$$

6. $$\begin{aligned} y &= 2\ln(x^3) \\ &= 6\ln x \\ \frac{dy}{dx} &= \frac{6}{x} \end{aligned}$$

7. $$\begin{aligned} \frac{dy}{dx} &= \frac{\cos x}{(} - \sin x) \\ &= -\tan x \end{aligned}$$

8. $$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sin 2x}(\cos 2x)(2) \\ &= \frac{2}{\tan 2x} \end{aligned}$$

9. $$\begin{aligned} y &= \ln(2\sqrt{x}) \\ &= \ln 2 + \frac{1}{2}\ln x \\ \frac{dy}{dx} &= \frac{1}{2x} \end{aligned}$$

10. $$\begin{aligned} \frac{dy}{dx} &= \ln x + x\left(\frac{1}{x}\right) \\ &= \ln x + 1 \end{aligned}$$

11. $$\begin{aligned} \frac{dy}{dx} &= 2\log_e x\left(\frac{1}{x}\right) \\ &= \frac{2\log_e x}{x} \end{aligned}$$

12. $$\begin{aligned} \frac{dy}{dx} &= 2x\ln x + x^2\left(\frac{1}{x}\right) \\ &= x(2\ln x + 1) \end{aligned}$$

13. $$\begin{aligned} \frac{dy}{dx} &= 2(3 + \ln x)\left(\frac{1}{x}\right) \\ &= \frac{6 + 2\ln x}{x} \end{aligned}$$

14. $$\begin{aligned} \frac{dy}{dx} &= -\frac{2}{x^2} + \frac{2}{x} \\ &= \frac{2}{x} - \frac{2}{x^2} \end{aligned}$$

15. $$\begin{aligned} y &= \ln\left(\frac{2}{x}\right) \\ &= \ln 2 - \ln x \\ \frac{dy}{dx} &= -\frac{1}{x} \end{aligned}$$

16. $$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{(\ln x)^2}\left(\frac{1}{x}\right) \\ &= -\frac{1}{x(\ln x)^2} \end{aligned}$$

17. $$\begin{aligned} \frac{dy}{dx} &= \frac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} \\ &= \frac{\ln x - 1}{(\ln x)^2} \\ &= \frac{1}{\ln x} - \frac{1}{(\ln x)^2} \end{aligned}$$

18. $$\begin{aligned} y &= \log_e\left[(x^2+1)^3\right] \\ &= 3\log_e(x^2+1) \\ \frac{dy}{dx} &= \frac{3}{x^2+1}(2x) \\ &= \frac{6x}{x^2+1} \end{aligned}$$

19. $$\begin{aligned} y &= \ln\left[\frac{(x-1)^3}{x+1}\right] \\ &= 3\ln(x-1) - \ln(x+1) \\ \frac{dy}{dx} &= \frac{3}{x-1} - \frac{1}{x+1} \end{aligned}$$

20. $$\begin{aligned} y &= \log_5 x \\ &= \frac{\ln x}{\ln 5} \\ \frac{dy}{dx} &= \frac{1}{x\ln 5} \end{aligned}$$

21. $$\begin{aligned} y &= \log_7 x \\ &= \frac{\ln x}{\ln 7} \\ \frac{dy}{dx} &= \frac{1}{x\ln 7} \end{aligned}$$

22. $$\frac{dy}{dx} = \frac{3}{x}$$
at $(e, 3)$: $$\frac{dy}{dx} = \frac{3}{e}$$

23. $$\frac{dy}{dx} = \ln x + 1 \quad \text{(see q.10)}$$
at (e, e): $$\begin{aligned} \frac{dy}{dx} &= \ln e + 1 \\ &= 2 \end{aligned}$$

24. (a)

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1+\frac{1}{x}\\ 1.5 &= 1+\frac{1}{x}\\ \frac{1}{x} &= 0.5\\ x &= 2\\ y &= 2+\ln(2\times 2)\end{aligned}$$

coordinates are $(2, 2+\ln 4)$

(b)

$$\begin{aligned}y &= \ln x+\ln(x+3)\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x}+\frac{1}{x+3}\\ &= \frac{2x+3}{x^2+3x}\\ 0.5 &= \frac{2x+3}{x^2+3x}\\ 1 &= \frac{4x+6}{x^2+3x}\\ x^2+3x &= 4x+6\\ x^2-x-6 &= 0\\ (x-3)(x+2) &= 0\\ x &= 3\\ \text{or } x &= -2\\ y &= \ln 18\\ \text{or } y &= \ln -2\end{aligned}$$

coordinates are $(3, \ln 18)$
(rejecting the second solution because $ln-2$ is not a real number.)

25.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x}\\ &= \frac{1}{\mathrm{e}^2}\\ y-y_1 &= m(x-x_1)\\ y-2 &= \frac{1}{\mathrm{e}^2}(x-e^2)\\ y-2 &= \frac{x}{\mathrm{e}^2}-1\\ y &= \frac{x}{\mathrm{e}^2}+1\end{aligned}$$

26.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sin x}\cos x\\ &= \frac{1}{\tan x}\\ y-y_1 &= m(x-x_1)\\ y-0 &= \frac{1}{\tan\frac{\pi}{6}}\left(x-\frac{\pi}{6}\right)\\ y &= \sqrt{3}\left(x-\frac{\pi}{6}\right)\end{aligned}$$

27. (a)

$$\begin{aligned}2x+6y\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}x}\ln x+\frac{y}{x}\\ \frac{\mathrm{d}y}{\mathrm{d}x}(6y-\ln x) &= \frac{y}{x}-2x\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{y}{x}-2x}{6y-\ln x}\\ &= \frac{y-2x^2}{x(6y-\ln x)}\end{aligned}$$

(b)

$$\begin{aligned}5+\frac{3}{2y+1}\left(2\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 3y+3x\frac{\mathrm{d}y}{\mathrm{d}x}\\ \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{6}{2y+1}-3x\right) &= 3y-5\\ \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{6-(6xy+3x)}{2y+1}\right) &= 3y-5\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \left(\frac{2y+1}{6-6xy-3x}\right)(3y-5)\\ &= \frac{(2y+1)(3y-5)}{6-6xy-3x}\\ &= \frac{6y^2-7y-5}{6-6xy-3x}\end{aligned}$$

28. (a)

$$\begin{aligned}\ln y &= x\ln 2\\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln 2\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y\ln 2\\ &= 2^x\ln 2\end{aligned}$$

(b)

$$\begin{aligned}\ln y &= x\ln 4\\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln 4\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y\ln 4\\ &= 4^x\ln 4\end{aligned}$$

Exercise 8E

1–4 No working required: do these in a single step.

5. $$\text{Try } y = \ln|x^2+1|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2+1}$$
$$\therefore \int \frac{2x}{x^2+1}\,\mathrm{d}x = \ln|x^2+1|$$

6–8 No working required: integrate in a single step.

9. $$\text{Try } y = \ln|x^2-3|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2-3}$$
$$\therefore \int \frac{8x}{x^2-3}\,\mathrm{d}x = 4\ln|x^2-3| + c$$

10. $$\text{Try } y = \ln|5x-3|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{5x-3}$$
$$\therefore \int \frac{5}{5x-3}\,\mathrm{d}x = \ln|5x-3| + c$$

11. $$\text{Try } y = \ln|2x+1|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x+1}$$
$$\therefore \int \frac{10}{2x+1}\,\mathrm{d}x = 5\ln|2x+1| + c$$

12. $$\text{Try } y = \ln|x^2+1|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2+1}$$
$$\therefore \int \frac{6x}{x^2+1}\,\mathrm{d}x = 3\ln|x^2+1| + c$$

13. $$\text{Try } y = \ln|\cos x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sin x}{\cos x}$$
$$\therefore \int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\ln|\cos x| + c$$

14. $$\text{Try } y = \ln|\sin x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{\sin x}$$
$$\therefore \int \frac{\cos x}{\sin x}\,\mathrm{d}x = \ln|\sin x| + c$$

15. $$\text{Try } y = \ln|\cos 2x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\sin 2x}{\cos 2x}$$
$$\therefore \int \frac{\sin 2x}{\cos 2x}\,\mathrm{d}x = -\frac{1}{2}\ln|\cos 2x| + c$$

16. $$\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x$$
$$= -\ln|\cos x| + c$$

17. $$\text{Try } y = \ln|\cos 5x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-5\sin 5x}{\cos 5x}$$
$$= -5\tan 5x$$
$$\therefore \int \tan 5x\,\mathrm{d}x = -\frac{1}{5}\ln|\cos 5x| + c$$

18. $$\int 6\tan 2x\,\mathrm{d}x = 6\int \tan 2x\,\mathrm{d}x$$
$$= 6\left(-\frac{1}{2}\ln|\cos 2x|\right) + c$$
$$= -3\ln|\cos 2x| + c$$

19. $$\text{Try } y = \ln|\sin x + \cos x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x - \sin x}{\sin x + \cos x}$$
$$\therefore \int \frac{\sin x - \cos x}{\sin x + \cos x}\,\mathrm{d}x = -\ln|\sin x + \cos x| + c$$

20. $$\text{Try } y = \ln|4x + \sin 2x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4 + 2\cos 2x}{4x + \sin 2x}$$
$$\therefore \int \frac{2 + \cos 2x}{4x + \sin 2x}\,\mathrm{d}x = \frac{1}{2}\ln|4x + \sin 2x| + c$$

21. $$\text{Try } y = \ln|\mathrm{e}^x + x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x + 1}{\mathrm{e}^x + x}$$
$$\therefore \int \frac{\mathrm{e}^x + 1}{\mathrm{e}^x + x}\,\mathrm{d}x = \ln|\mathrm{e}^x + x| + c$$

22. $$\int_1^3 \frac{1}{x}\,\mathrm{d}x = \left[\ln|x|\right]_1^3$$
$$= \ln 3 - \ln 1$$
$$= \ln 3$$

23. $$\int_{-3}^{-2} \frac{3}{x}\,\mathrm{d}x = \left[3\ln|x|\right]_{-3}^{-2}$$
$$= 3\ln 2 - 3\ln 3)$$
$$= 3\ln\frac{2}{3}$$
(Make sure you understand how this is equivalent to the answer Sadler gives.)

24. $$\int_1^2 \left(\mathrm{e}^x + \frac{1}{x}\right)\mathrm{d}x = \left[\mathrm{e}^x + \ln|x|\right]_1^2$$
$$= (\mathrm{e}^2 + \ln 2) - (\mathrm{e} + \ln 1)$$
$$= \mathrm{e}^2 - \mathrm{e} + \ln 2$$

25. $$\begin{aligned} \text{Try } y &= 4^x \\ \text{then } \ln y &= x\ln 4 \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln 4 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y\ln 4 \\ &= 4^x \ln 4 \\ \therefore \quad \int 4^x\,\mathrm{d}x &= \frac{4^x}{\ln 4} + c \end{aligned}$$

26. $$\begin{aligned} \text{Area} &= \int_1^3 \left|\frac{2x+1}{x}\right|\mathrm{d}x \\ &= \int_1^3 \left|2 + \frac{1}{x}\right|\mathrm{d}x \\ &= [2x + \ln x]_1^3 \\ &= (6 + \ln 3) - (2 + \ln 1) \\ &= 4 + \ln 3 \text{ square units} \end{aligned}$$

(Remember to use your CAS tool to answer questions like this in calculator-assumed assessments.)

27. One bound for the definite integral will be the y−axis, i.e. $x = 0$. To find the other,

$$\begin{aligned} \frac{1}{x+2} - 1 &= 0 \\ x &= -1 \\ \text{Area} &= \int_{-1}^0 \left|\frac{1}{x+2} - 1\right|\mathrm{d}x \\ &= \int_{-1}^0 1 - \frac{1}{x+2}\,\mathrm{d}x \\ &= [x - \ln|x+2|]_{-1}^0 \\ &= (0 - \ln 2) - (-1 - \ln 1) \\ &= 1 - \ln 2 \end{aligned}$$

28. $$\begin{aligned} \text{Area} &= \int_0^{\frac{\pi}{6}} |\tan x|\,\mathrm{d}x \\ &= [-\ln|\cos x|]_0^{\frac{\pi}{6}} \\ &= (-\ln\cos\frac{\pi}{6}) - (-\ln\cos 0) \\ &= -\ln\frac{\sqrt{3}}{2} + \ln 1 \\ &= -\left(\frac{\ln 3}{2} - \ln 2\right) \\ &= \ln 2 - \frac{\ln 3}{2} \end{aligned}$$

29. $$\begin{aligned} \frac{a}{x+4} + \frac{b}{x+2} &= \frac{a(x+2) + b(x+4)}{(x+4)(x+2)} \\ &= \frac{(a+b)x + 2a + 4b}{(x+4)(x+2)} \\ \therefore \quad (a+b)x + 2(a+2b) &= 2(4x+13) \\ a + b &= 8 \\ \text{and } a + 2b &= 13 \\ \therefore \quad b &= 5 \\ \text{and } a &= 3 \\ \therefore \quad \int \frac{2(4x+13)}{(x+4)(x+2)}\,\mathrm{d}x &= \int \left(\frac{3}{x+4} + \frac{5}{x+2}\right)\mathrm{d}x \\ &= 3\ln|x+4| + 5\ln|x+2| + c \end{aligned}$$

30. (a) $$\begin{aligned} \int_1^k \frac{2}{x}\,\mathrm{d}x &= 1 \\ [2\ln|x|]_1^k &= 1 \\ 2\ln k - 2\ln 1 &= 1 \\ 2\ln k &= 1 \\ \ln k &= \frac{1}{2} \\ k &= \mathrm{e}^{\frac{1}{2}} \end{aligned}$$

(b) $$\begin{aligned} \int_1^b \frac{2}{x}\,\mathrm{d}x &= 0.5 \\ [2\ln|x|]_1^b &= 0.5 \\ 2\ln b - 2\ln 1 &= 0.5 \\ 2\ln b &= 0.5 \\ \ln b &= 0.25 \\ b &= \mathrm{e}^{0.25} \end{aligned}$$

(c) $$\begin{aligned} c &= \frac{1+k}{2} \\ &= \frac{1+e^{0.5}}{2} \\ \int_1^c \frac{2}{x}\,\mathrm{d}x &= [2\ln|x|]_1^c \\ &= 2\ln c - 2\ln 0 \\ &= 2\ln c \\ &= 2\ln\frac{1+e^{0.5}}{2} \\ &= 2\ln(1+e^{0.5}) - 2\ln 2 \\ &= 2\ln(1+\mathrm{e}^{0.5}) - \ln 4 \end{aligned}$$

Miscellaneous Exercise 8

1. (a)
$$\begin{aligned} r &= 5\sqrt{\sqrt{3})^2 + 1^2} \\ &= 3 \\ \tan\theta &= \frac{1}{\sqrt{3}} \\ \theta &= \frac{\pi}{6} + n\pi \quad (3^{\text{rd}} \text{ quadrant.}) \\ \theta &= -\frac{5\pi}{6} \\ -5(\sqrt{3}+\mathrm{i}) &= 3\,\mathrm{cis} -\frac{5\pi}{6} \end{aligned}$$

(b)
$$\begin{aligned} a &= 6\cos\frac{3\pi}{4} \\ &= 6\left(-\frac{\sqrt{2}}{2}\right) \\ &= -3\sqrt{2} \\ b &= 6\sin\frac{3\pi}{4} \\ &= 6\left(\frac{\sqrt{2}}{2}\right) \\ &= 3\sqrt{2} \\ 6\,\mathrm{cis}\,\frac{3\pi}{4} &= -3\sqrt{2} + 3\sqrt{2}\mathrm{i} \end{aligned}$$

2. (a) Do in a single step. No working needed.

(b) Do in a single step. No working needed.

(c) Do in a single step. No working needed.

(d)
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(5-3x)-(2x+3)(-3)}{(5-3x)^2} \\ &= \frac{10-6x+6x+9}{(5-3x)^2} \\ &= \frac{19}{(5-3x)^2} \end{aligned}$$

(e)
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 3(2x+3)^2(2) \\ &= 6(2x+3)^2 \end{aligned}$$

(f)
$$\begin{aligned} 2y + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 3\cos x \\ \frac{\mathrm{d}y}{\mathrm{d}x}(2x+2y) &= 3\cos x - 2y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3\cos x - 2y}{2(x+y)} \end{aligned}$$

(g)
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{6\cos 3t}{10\sin 2t} \\ &= \frac{3\cos 3t}{5\sin 2t} \end{aligned}$$

3. (a)
$$\begin{aligned} \int \sin^3 x\,\mathrm{d}x &= \int \sin x\sin^2 x\,\mathrm{d}x \\ &= \int \sin x(1-\cos^2 x)\,\mathrm{d}x \\ &= \int \sin x\,\mathrm{d}x - \int \sin x\cos^2 x\,\mathrm{d}x \\ &= -\cos x + \frac{\cos^3 x}{3} + c \end{aligned}$$

(b)
$$\begin{aligned} \int 2x^7(1+x)\,\mathrm{d}x &= \int (2x^7 + 2x^8)\,\mathrm{d}x \\ &= \frac{2x^8}{8} + \frac{2x^9}{9} + c \\ &= \frac{x^8}{4} + \frac{2x^9}{9} + c \\ &= \frac{x^8}{3}6(9+8x) + c \end{aligned}$$

(c) Let $u = 1 + x,\ x = u - 1,\ \mathrm{d}u = \mathrm{d}x$
$$\begin{aligned} \int 2x(1+x)^7\,\mathrm{d}x &= \int 2u^7(u-1)\,\mathrm{d}u \\ &= \int 2u^8 - 2u^7\,\mathrm{d}u \\ &= \frac{2u^9}{9} - \frac{u^8}{4} + c \\ &= \frac{u^8}{3}6(8u-9) + c \\ &= \frac{(x+1)^8}{3}6(8(x+1)-9) + c \\ &= \frac{(x+1)^8}{3}6(8x-1) + c \end{aligned}$$

(d) Observe that we have a multiple of $\mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$
Guess $y = \mathrm{e}^{x^2+5}$,
then $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\mathrm{e}^{x^2+5}$,
hence $\int 6x\mathrm{e}^{x^2+5}\,\mathrm{d}x = 3\mathrm{e}^{x^2+5} + c$

4. (a)
$$\begin{aligned} \int \frac{x^2+1}{x}\,\mathrm{d}x &= \int x + \frac{1}{x}\,\mathrm{d}x \\ &= 0.5x^2 + \ln x + c \end{aligned}$$

(b) Observe that we have a multiple of $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$
Guess $y = \ln(x^2+1)$,
then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2+1}$,
hence $\int \frac{x}{x^2+1}\,\mathrm{d}x = 0.5\ln(x^2+1) + c$

(c) Observe that we have a multiple of $\mathrm{f}'(x)(\mathrm{f}(x))^n$ for $n = -\frac{1}{2}$
Guess $y = \sqrt{x^2+1}$,
then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\sqrt{x^2+1}}$,
hence $\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x = \sqrt{x^2+1} + c$

(d)

5. You should recognise this limit as having the form of a first-principles differentiation.
$$\begin{aligned} \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \end{aligned}$$

6.
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\
\mathrm{e}^x\cos x - \mathrm{e}^x \sin x &= 0\\
\mathrm{e}^x(\cos x - \sin x) &= 0\\
\cos x &= \sin x\\
\tan x &= 1\\
x &= \frac{\pi}{4} + n\pi\\
x &\in \left\{-\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}\right\}
\end{aligned}$$

7.
$$\begin{aligned}
\overrightarrow{\mathrm{OP}} &= \overrightarrow{\mathrm{OA}} + \frac{4}{5}\overrightarrow{\mathrm{AB}}\\
&= \begin{pmatrix}1\\6\\-7\end{pmatrix} + \frac{4}{5}\left(\begin{pmatrix}-4\\1\\3\end{pmatrix} - \begin{pmatrix}1\\6\\-7\end{pmatrix}\right)\\
&= \begin{pmatrix}1\\6\\-7\end{pmatrix} + \frac{4}{5}\begin{pmatrix}-5\\-5\\10\end{pmatrix}\\
&= \begin{pmatrix}1\\6\\-7\end{pmatrix} + \begin{pmatrix}-4\\-4\\8\end{pmatrix}\\
&= \begin{pmatrix}-3\\2\\1\end{pmatrix}
\end{aligned}$$

8. (a)
$$\begin{aligned}
|\mathbf{a}| &= \sqrt{1^2+1^2+3^2}\\
&= \sqrt{11}\\
|\mathbf{b}| &= \sqrt{2^2+1^2+1^2}\\
&= \sqrt{6}\\
\cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\\
&= \frac{1\times 2 + -1\times -1 + 3\times 1}{\sqrt{11}\sqrt{6}}\\
&= \frac{6}{\sqrt{11}\sqrt{6}}\\
\theta &\approx 42^\circ
\end{aligned}$$

(b) Since we know the magnitude of $\mathbf{a}$ the two unit vectors parallel to $\mathbf{a}$ can be obtained without any further working.

(c) Any of the following will yield a suitable equation:

- $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{a} - \mathbf{b})$
- $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$
- $\mathbf{r} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b})$
- $\mathbf{r} = \mathbf{b} + \lambda(\mathbf{b} - \mathbf{a})$

Simply substitute the values for **a** and **b** into one of the above.

(d) We have a point on the plane and a perpendicular or *normal*, so we use the normal form for the equation of a plane:

$$\begin{aligned}
\mathbf{r}\cdot\overrightarrow{\mathrm{AB}} &= \mathbf{a}\cdot\overrightarrow{\mathrm{AB}}\\
\mathbf{r}\cdot(\mathbf{b}-\mathbf{a}) &= \mathbf{a}\cdot(\mathbf{b}-\mathbf{a})\\
\mathbf{r}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix} &= \begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}\\
\mathbf{r}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix} &= -5
\end{aligned}$$

9. L.H.S.:
$$\begin{aligned}
&(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)\\
&= \cos^2\theta - \sin^2\theta\\
&= (1-\sin^2\theta) - \sin^2\theta\\
&= 1 - 2\sin^2\theta\\
&= \text{R.H.S.}
\end{aligned}$$

□

10. L.H.S.:
$$\begin{aligned}
\cos 3x &= \cos(x+2x)\\
&= \cos x\cos 2x - \sin x \sin 2x\\
&= \cos x(2\cos^2 x - 1) - \sin x(2\sin x\cos x)\\
&= 2\cos^3 x - \cos x - 2\sin^2 x\cos x\\
&= 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x)\\
&= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x\\
&= 4\cos^3 x - 3\cos x\\
&= \text{R.H.S.}
\end{aligned}$$

□

11.
$$\begin{aligned}
xy &= \cos x\\
y + x\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin x\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-\sin x - y}{x}\\
&= \frac{-\sin\left(\frac{\pi}{3}\right) - \frac{3}{2\pi}}{\frac{\pi}{3}}\\
&= \left(-\sqrt{3}2 - \frac{3}{2\pi}\right)\times\frac{3}{\pi}\\
&= -3\sqrt{3}2\pi - \frac{9}{2\pi^2}\\
&= -\frac{3(\sqrt{3}\pi + 3)}{2\pi^2}
\end{aligned}$$

12.
$$\begin{aligned}
x^3 + 2x^2y + y^3 &= 10\\
3x^2 + 4xy + 2x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\
(2x^2+3y^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= -3x^2 - 4xy\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x(3x+4y)}{2x^2+3y^2}
\end{aligned}$$

Note that although the question asks for $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of x, the answer provided is a function of x and y. You are *not* expected to be able to eliminate y from this expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. I believe the question is in error and should not specify "as a function of x".

13. (a)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x^2\sin 2x}\left((2x)(\sin 2x)+(x^2)(2\cos 2x)\right)\\ &= \frac{2x\sin 2x+2x^2\cos 2x}{x^2\sin 2x}\\ &= \frac{2\sin 2x+2x\cos 2x}{x\sin 2x}\\ &= \frac{2}{x}+\frac{2}{\tan 2x}\end{aligned}$$

(b)
$$\begin{aligned}y &= \frac{\ln x}{\ln 2}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x\ln 2}\end{aligned}$$

(c)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4\mathrm{e}^{4x}(3x\sin^4 x)-\mathrm{e}^{4x}(3\sin^4 x+3x(4\sin^3 x\cos x))}{(3x\sin^4 x)^2}\\ &= \frac{12x\mathrm{e}^{4x}\sin^4 x-(3\mathrm{e}^{4x}\sin^4 x+12x\mathrm{e}^{4x}\sin^3 x\cos x)}{(3x\sin^4 x)^2}\\ &= \frac{12x\mathrm{e}^{4x}\sin^4 x-3\mathrm{e}^{4x}\sin^4 x-12x\mathrm{e}^{4x}\sin^3 x\cos x}{(3x\sin^4 x)^2}\\ &= \frac{3\mathrm{e}^{4x}\sin^3 x(4x\sin x-\sin x-4x\cos x)}{9x^2\sin^8 x}\\ &= \frac{\mathrm{e}^{4x}(\sin x(4x-1)-4x\cos x)}{3x^2\sin^5 x}\end{aligned}$$

14.
$$\begin{aligned}2x+2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y}\\ &= -\frac{3}{4}\\ y-y_1 &= m(x-x_1)\\ y-4 &= -\frac{3}{4}(x-3)\\ 4(y-4) &= -3(x-3)\\ 4y-16 &= -3x+9\\ 4y &= -3y+25\\ 3x+4y &= 25\end{aligned}$$

15. When $x=1$,

$$\begin{aligned}y^2+5y+1 &= 15\\ y^2+5y-14 &= 0\\ (y+7)(y-2) &= 0\\ y &= -7\\ \text{or } y &= 2\end{aligned}$$

$$\begin{aligned}\text{Then } 2y\frac{\mathrm{d}y}{\mathrm{d}x}+5y+5x\frac{\mathrm{d}y}{\mathrm{d}x}+2x &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x}(2y+5x) &= -(2x+5y)\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2x+5y}{5x+2y}\end{aligned}$$

At $(1,-7)$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2-35}{5-14}\\ &= -\frac{33}{9}\\ &= -\frac{11}{3}\\ y+7 &= -\frac{11}{3}(x-1)\\ 3(y+7)+11(x-1) &= 0\\ 3y+11x &= -10\end{aligned}$$

At $(1,2)$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2+10}{5+4}\\ &= -\frac{12}{9}\\ &= -\frac{4}{3}\\ y-2 &= -\frac{4}{3}(x-1)\\ 3(y-2) &= -4(x-1)\\ 4x+3y &= 10\end{aligned}$$

16. First find find $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}\\ &= \frac{1+\frac{1}{t^2}}{2t-6}\end{aligned}$$

The curve cuts the $y-$axis when $x=0$

$$\begin{aligned}t^2-6t+5 &= 0\\ (t-1)(t-5) &= 0\\ t &= 1\\ \text{or } t &= 5\end{aligned}$$

At $t=1$:

$$\begin{aligned}y &= 1-\frac{1}{1}\\ &= 0\\ \text{and } \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1+\frac{1}{1}}{2-6}\\ &= -\frac{2}{4}\\ &= -0.5\end{aligned}$$

giving the tangent line

$$\begin{aligned}y-0 &= -0.5(x-0)\\ y &= -0.5x\end{aligned}$$

At $t = 5$:

$$\begin{aligned} y &= 5 - \frac{1}{5} \\ &= 4.8 \\ \text{and } \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1 + \frac{1}{25}}{10 - 6} \\ &= \frac{1.04}{4} \\ &= 0.26 \end{aligned}$$

giving the tangent line

$$\begin{aligned} y - 4.8 &= 0.26(x - 0) \\ y &= 0.26x + 4.8 \end{aligned}$$

17. (a)
$$\begin{aligned} 1 + \frac{1}{y} &= x \\ -\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y^2 \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2y\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= 2y(y^2) \\ &= 2y^3 \end{aligned}$$

(b)
$$\begin{aligned} y^2 - \frac{5}{y} &= x \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{5}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x}\left(2y + \frac{5}{y^2}\right) &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{2y^3 + 5}{y^2}\right) &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y^2}{2y^3 + 5} \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{2y\frac{\mathrm{d}y}{\mathrm{d}x}(2y^3 + 5) - y^2(6y^2\frac{\mathrm{d}y}{\mathrm{d}x})}{(2y^3 + 5)^2} \\ &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{4y^4 + 10y - 6y^4}{(2y^3 + 5)^2}\right) \\ &= \frac{y^2}{2y^3 + 5} \times \frac{10y - 2y^4}{(2y^3 + 5)^2} \\ &= \frac{10y^3 - 2y^6}{(2y^3 + 5)^3} \\ &= \frac{2y^3(5 - y^3)}{(2y^3 + 5)^3} \end{aligned}$$

18. (a) For $0 < k < \frac{\pi}{2}$ (i.e. first quadrant) y is positive so the area between the curve and the x−axis is equal to the definite integral:

$$\begin{aligned} A &= \int_0^k 3\sin^2 x \cos x \,\mathrm{d}x \\ &= \left[\sin^3 x\right]_0^k \\ &= \sin^3 k - \sin^3 0 \\ &= (\sin^3 k) \text{ units}^2 \end{aligned}$$

(b) For $\frac{\pi}{2} < k < \pi$ (i.e. second quadrant) as x goes from 0 to k the value of y has a positive component and a negative component so so the area between the curve and the x−axis must be taken piecewise. (If you were doing this numerically with a calculator it would be simplest to take the definite integral of the absolute value of y.)

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} 3\sin^2 x \cos x \,\mathrm{d}x - \int_{\frac{\pi}{2}}^k 3\sin^2 x \cos x \,\mathrm{d}x \\ &= \left[\sin^3 x\right]_0^{\frac{\pi}{2}} - \left[\sin^3 x\right]_{\frac{\pi}{2}}^k \\ &= \left(\sin^3 \frac{\pi}{2} - \sin^3 0\right) - \left(\sin^3 k - \sin^3 \frac{\pi}{2}\right) \\ &= (1 - 0) - (\sin^3 k - 1) \\ &= (2 - \sin^3 k) \text{ units}^2 \end{aligned}$$

19. (a)
$$\begin{aligned} y &= 0 \\ (\log_e x)^2 - 1 &= 0 \\ (\log_e x)^2 &= 1 \\ \log_e x &= \pm 1 \\ x &= e^{\pm 1} \\ &\text{A} : (e^{-1}, 0) \\ &\text{B} : (e, 0) \end{aligned}$$

Having found all the roots will suffice to prove that the function does not cut the x−axis at any other point. For the y−axis: Suppose that a point exists where the function cuts the y− axis. Then the function is defined for $x = 0$. But $\log_e 0$ is undefined: a contradiction. Therefore there is no point where the function cuts the y−axis. □

(b) At stationary points the first derivative is zero.

$$\begin{aligned} \frac{2\log_e x}{x} &= 0 \\ \log_e x &= 0 \\ x &= 1 \end{aligned}$$

That this has only solution is sufficient proof that there are no other stationary points.

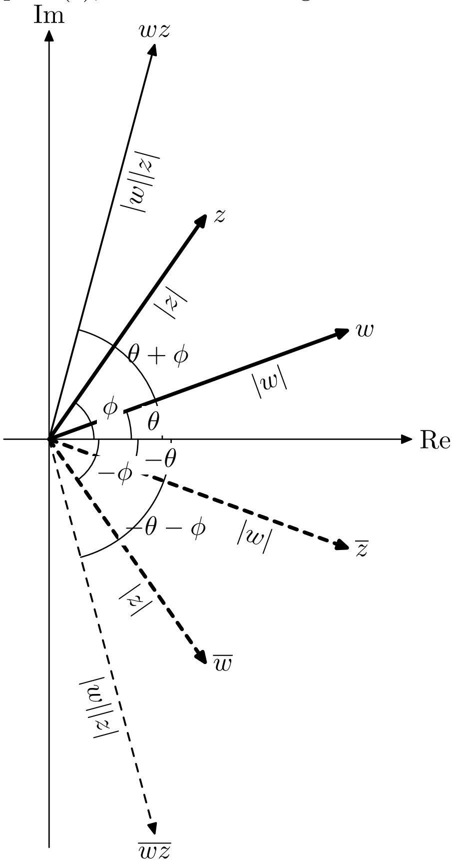
$$\begin{aligned} y &= (\log_e 1) - 1 \\ &= -1 \\ &\text{C} : (1, -1) \end{aligned}$$

(c) At points of inflection the second derivative is zero.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{2\log_e x}{x} &= 0\\ \frac{2-2\log_e x}{x^2} &= 0\\ 1-\log_e x &= 0\\ \log_e x &= 1\\ x &= \mathrm{e}\end{aligned}$$

Therefore there is one point of inflection at point B:$(\mathrm{e}, 0)$.

20. Refer to Sadler's answers for (a) and (b). For part (c), consider this diagram:



Let $\theta = \arg(w)$

$\phi = \arg(z)$

A complex conjugate has the same modulus and opposite argument.

Then $\arg(\bar{w}) = -\theta$

$\arg(\bar{z}) = -\phi$

and $|\bar{w}| = |w|$

$|\bar{z}| = |z|$

We obtain the products by multiplying the moduli and adding the arguments:

$$\begin{aligned}|wz| &= |w||z|\\ |\overline{wz}| &= |wz|\\ &= |w||z|\\ &= |\bar{w}||\bar{z}|\\ &= |\bar{w}\bar{z}|\\ \arg(wz) &= \theta+\phi\\ \arg(\overline{wz}) &= -\arg(wz)\\ &= -(\theta+\phi)\\ &= (-\theta)+(-\phi)\\ &= \arg(\bar{w})+\arg(\bar{z})\\ &= \arg(\bar{w}\bar{z})\end{aligned}$$

Thus by comparing modulus and argument we can see that $\overline{wz} = \bar{w}\bar{z}$ as required.

21. $$\begin{aligned}\frac{ax}{x^2-1}+\frac{b}{x+1} &= \frac{ax}{(x-1)(x+1)}+\frac{b}{x+1}\\ &= \frac{ax+b(x-1)}{x^2-1}\\ &= \frac{ax+bx-b}{x^2-1}\\ &= \frac{(a+b)x-b}{x^2-1}\end{aligned}$$

$$\begin{aligned}\therefore\quad b &= 5\\ a+b &= 7\\ a &= 2\end{aligned}$$

$$\begin{aligned}\int\frac{7x-5}{x^2-1}\,\mathrm{d}x &= \int\left(\frac{2x}{x^2-1}+\frac{5}{x+1}\right)\mathrm{d}x\\ &= \ln|x^2-1|+5\ln|x+1|+c\end{aligned}$$

22. The non-zero point where graphs intersect is given by

$$\begin{aligned}\sin^2 x &= \sin x\cos x\\ \sin x &= \cos x\end{aligned}$$

(where $\sin x = 0$ gives the points where they intersect on the $x-$axis.)

$$x = \frac{\pi}{4}$$

(a) $$\begin{aligned}A &= \int_0^{\frac{\pi}{4}}(\sin x\cos x-\sin^2 x)\,\mathrm{d}x\\ &= \int_0^{\frac{\pi}{4}}(\frac{1}{2}\sin 2x+\frac{1}{2}-\sin^2 x-\frac{1}{2})\,\mathrm{d}x\\ &= \int_0^{\frac{\pi}{4}}(\frac{1}{2}\sin 2x+\frac{1}{2}(1-2\sin^2 x)-\frac{1}{2})\,\mathrm{d}x\\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}}(\sin 2x+\cos 2x-1)\,\mathrm{d}x\\ &= \frac{1}{4}\int_0^{\frac{\pi}{4}}(2\sin 2x+2\cos 2x-2)\,\mathrm{d}x\\ &= \frac{1}{4}[-\cos 2x+\sin 2x-2x]_0^{\frac{\pi}{4}}\\ &= \frac{1}{4}\left((-\cos\frac{\pi}{2}+\sin\frac{\pi}{2}-\frac{\pi}{2})-(-\cos 0+\sin 0-0)\right)\\ &= \frac{1}{4}\left((0+1-\frac{\pi}{2})-(-1+0)\right)\\ &= \frac{4-\pi}{8}\text{ units}^2\end{aligned}$$

(b)
$$\begin{aligned}A &= \int_0^{\pi} |\sin x\cos x - \sin^2 x|\,\mathrm{d}x\\ &= \frac{4-\pi}{8} + \int_{\frac{\pi}{4}}^{\pi}(\sin^2 x - \sin x\cos x)\,\mathrm{d}x\\ &= \frac{4-\pi}{8} + \frac{1}{4}\left[\cos 2x - \sin 2x + 2x\right]_{\frac{\pi}{4}}^{\pi}\\ &= \frac{4-\pi}{8} + \frac{1}{4}\Big((\cos 2\pi - \sin 2\pi + 2\pi) - (\cos\frac{\pi}{2} - \sin\frac{\pi}{2} + \frac{\pi}{2})\Big)\\ &= \frac{4-\pi}{8} + \frac{1}{4}\Big((1-0+2\pi) - (0-1+\frac{\pi}{2})\Big)\\ &= \frac{4-\pi}{8} + \frac{1}{4}\left(2 + \frac{3\pi}{2}\right)\\ &= \frac{4-\pi}{8} + \frac{4+3\pi}{8}\\ &= \frac{8+2\pi}{8}\\ &= \frac{4+\pi}{4}\ \text{units}^2\end{aligned}$$

23.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \sqrt{3}\cos x - \sin x &= 0\\ \sqrt{3}\cos x &= \sin x\\ \tan x &= \sqrt{3}\\ x &\in \left\{-\frac{5\pi}{3}, -\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}\right\}\end{aligned}$$

and substituting to obtain values for y gives the coordinates

$$(-\frac{5\pi}{3}, 2); (-\frac{2\pi}{3}, -2); (\frac{\pi}{3}, 2); (\frac{4\pi}{3}, -2)$$

Now using the second derivative test

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\sqrt{3}\sin x - \cos x\\ &= -y\end{aligned}$$

Hence $(-\frac{5\pi}{3}, 2)$ and $(\frac{\pi}{3}, 2)$ have negative second derivative are so local maxima and similarly $(-\frac{2\pi}{3}, -2)$ and $(\frac{4\pi}{3}, -2)$ are local minima.

24. (a) If the normal of plane A is parallel to plane B then planes A and B are perpendicular.
If the normal of plane A is perpendicular to the normal of plane B then the normal of plane A is parallel to plane B and the planes are perpendicular.
Thus we can test for perpendicularlity of planes by testing for perpendicularity of their respective normals.

$$(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 0$$

∴ the first and second planes are perpendicular.

$$(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = 0$$

∴ the first and third planes are perpendicular.

$$(4\mathbf{i} - \mathbf{j} + 7\mathbf{k}) \cdot (-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = 0$$

∴ the second and third planes are perpendicular.

□

(b)
$$\begin{aligned}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k}) &= 4\\ 2a + b - c &= 4\\ (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} + 7\mathbf{k}) &= 8\\ 4a - b + 7c &= 8\\ (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) &= 9\\ -a + 3b + c &= 9\end{aligned}$$

(c) You should be using your calculator for this, but if you wish to see how it would be done without, use the elimination method just as you would for solving two equations in two unknowns.

$$\begin{aligned}2a + b - c &= 4 && ①\\ 4a - b + 7c &= 8 && ②\\ -a + 3b + c &= 9 && ③\\ 7b + c &= 22 && ①+2\times③\rightarrow④\\ -3b + 9c &= 0 && -2\times①+②\rightarrow⑤\\ 22c &= 22 && (④+\tfrac{7}{3}⑤)\\ c &= 1\\ 7b + 1 &= 22\\ b &= 3\\ 2a + 3 - 1 &= 4\\ a &= 1\end{aligned}$$

Thus the point that lies on all three planes is $(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$.

25. Refer to Sadler page 87.

26. (a) To prove: $\operatorname{cis} 0 = 1$
Proof:

$$\begin{aligned}\text{L.H.S.} &= \operatorname{cis} 0\\ &= \cos 0 + \mathrm{i}\sin 0\\ &= 1 + 0\mathrm{i}\\ &= 1\\ &= \text{R.H.S.}\end{aligned}$$

□

(b) To prove: $\operatorname{cis}\alpha\operatorname{cis}\beta = \operatorname{cis}(\alpha + \beta)$
Proof:

$$\begin{aligned}\text{L.H.S.} &= \operatorname{cis}\alpha\operatorname{cis}\beta\\ &= (\cos\alpha + \mathrm{i}\sin\alpha)(\cos\beta + \mathrm{i}\sin\beta)\\ &= \cos\alpha\cos\beta + \mathrm{i}\cos\alpha\sin\beta\\ &\quad + \mathrm{i}\sin\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &\quad + \mathrm{i}(\cos\alpha\sin\beta + \sin\alpha\cos\beta)\\ &= \cos(\alpha + \beta) + \mathrm{i}\sin(\alpha + \beta)\\ &= \operatorname{cis}(\alpha + \beta)\\ &= \text{R.H.S.}\end{aligned}$$

□

(c) To prove: $\operatorname{cis}(-\alpha) = (\operatorname{cis}\alpha)^{-1}$
Proof:

$$\begin{aligned}
\text{R.H.S.} &= (\operatorname{cis}\alpha)^{-1}\\
&= \frac{1}{\cos\alpha + \mathrm{i}\sin\alpha}\\
&= \frac{\cos\alpha - \mathrm{i}\sin\alpha}{(\cos\alpha + \mathrm{i}\sin\alpha)(\cos\alpha - \mathrm{i}\sin\alpha)}\\
&= \frac{\cos(-\alpha) + \mathrm{i}\sin(-\alpha)}{\cos^2\alpha - \mathrm{i}^2\sin^2\alpha}\\
&= \frac{\operatorname{cis}(-\alpha)}{\cos^2\alpha + \sin^2\alpha}\\
&= \operatorname{cis}(-\alpha)\\
&= \text{L.H.S.}
\end{aligned}$$

□

27. For this question it is essential that you understand that the given set of points describes the locus of a circle on the Argand plane centred at $12 + 5\mathrm{i}$ and having a radius of 4.

(a) The minimum $\Im(z)$ occurs directly below the centre, i.e. $5 - 4 = 1$.

(b) The maximum and minimum $\Re(z)$ occur right and left of the centre, i.e. at 12 ± 4, so the maximum of $|\Re(z)|$ is $12 + 4 = 16$.

(c) The maximum $|z|$ is the furthest point on the circle away from the origin. Hence it is the point 4 units from the centre that lies along the line between the centre and the origin. The centre lies 13 units from the origin, so the maximum of $|z|$ is $13 + 4 = 17$

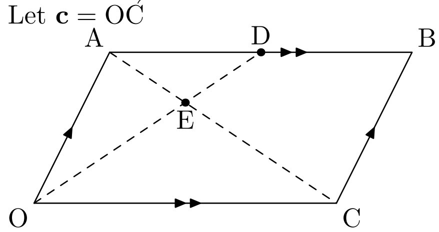
(d) The minimum possible $|z|$ is by similar reasoning $13 - 4 = 9$.

(e) Let A be the point on the circle with minimum $\arg(z)$.
Let C be the centre of the circle at $12 + 5\mathrm{i}$.
Let O be the origin at $0 + 0\mathrm{i}$.
It follows that OA is a tangent to the circle (since otherwise there would be a point on the circle below the line OA with a smaller argument).
Thus $\angle CAO$ is a right angle.

$$\begin{aligned}
CA &= 4\\
OC &= 13\\
\angle COA &= \sin^{-1}\frac{4}{13}\\
&= 0.3128\\
\arg(C) &= \tan^{-1}\frac{5}{12}\\
&= 0.3948\\
\therefore \quad \arg(A) &= 0.3948 - 0.3128\\
&= 0.08 \quad \text{(2d.p.)}
\end{aligned}$$

(f) By similar reasoning, the maximum $\arg z$ is $0.3948 + 0.3128 = 0.71$ (2d.p.).

28. Let D be the midpoint of AB.
Let E be the intersection between OD and AC.
Let $\mathbf{a} = \overrightarrow{\mathrm{OA}}$
Let $\mathbf{c} = \overrightarrow{\mathrm{OC}}$



To prove:

$$\overrightarrow{\mathrm{AE}} = \frac{1}{3}\overrightarrow{\mathrm{AC}}$$

Proof:

$$\begin{aligned}
\overrightarrow{\mathrm{OD}} &= \mathbf{a} + 0.5\mathbf{c}\\
\overrightarrow{\mathrm{AE}} &= p\overrightarrow{\mathrm{AC}}\\
&= p(-\mathbf{a} + \mathbf{c})\\
\overrightarrow{\mathrm{AE}} &= -\mathbf{a} + q\overrightarrow{\mathrm{OD}}\\
&= -\mathbf{a} + q(\mathbf{a} + 0.5\mathbf{c})\\
&= (q-1)\mathbf{a} + 0.5q\mathbf{c}\\
\therefore \quad p(-\mathbf{a} + \mathbf{c}) &= (q-1)\mathbf{a} + 0.5q\mathbf{c}
\end{aligned}$$

equating corresponding components

$$\begin{aligned}
p &= 1 - q\\
p &= 0.5q\\
q &= 2p\\
p &= 1 - 2p\\
3p &= 1\\
\therefore \quad \overrightarrow{\mathrm{AE}} &= \frac{1}{3}\overrightarrow{\mathrm{AC}}
\end{aligned}$$

□

Given the content of the course, a vector proof would be the anticipated approach for this question. However, since the question does not specify that you must use vector methods, you could take a purely geometric approach:

$$\begin{aligned}
\angle AED &\cong \angle OEC && \text{(vertically opposite)}\\
\angle EAB &\cong \angle ECO && \text{(alternate angles)}\\
\therefore \quad \triangle AED &\sim \triangle CEO && \text{(AAA)}\\
AD &= \frac{1}{2}AB && \text{(given)}\\
OC &= AB && \text{(opposite sides of a parallelogram)}\\
\therefore \quad AD &= \frac{1}{2}OC\\
\therefore \quad AE &= \frac{1}{2}EC && \text{(corresponding sides of similar } \triangle\text{s)}\\
\therefore \quad AE &= \frac{1}{3}AC
\end{aligned}$$

□

29. (a)

$$\begin{aligned}
\frac{dy}{dx} &= 3(2x+3)^2(2) \\
&= 6(2x+3)^2 \\
\frac{d^2y}{dx^2} &= 12(2x+3)(2) \\
&= 24(2x+3) \\
\frac{dx}{dy} &= \frac{1}{6}y^{-\frac{2}{3}} \\
\frac{d^2x}{dy^2} &= -\frac{1}{9}y^{-\frac{5}{3}} \\
&= -\frac{1}{9}\left((2x+3)^3\right)^{-\frac{5}{3}} \\
&= -\frac{1}{9}(2x+3)^{-5} \\
-\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3 &= \frac{1}{9}(2x+3)^{-5}\left(6(2x+3)^2\right)^3 \\
&= \frac{6^3}{9}(2x+3)^{-5}(2x+3)^6 \\
&= 24(2x+3) \\
&= \frac{d^2y}{dx^2}
\end{aligned}$$

(b) Start with $y = f(x)$ and differentiate with respect to y:

$$\begin{aligned}
y &= f(x) \\
1 &= \frac{dy}{dx}\frac{dx}{dy} \qquad \text{(chain rule)} \\
0 &= \left(\frac{d^2y}{dx^2}\frac{dx}{dy}\right)\frac{dx}{dy} + \frac{dy}{dx}\frac{d^2x}{dy^2} \\
\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^2 &= -\frac{dy}{dx}\frac{d^2x}{dy^2} \\
\frac{d^2y}{dx^2} &= \frac{-\frac{dy}{dx}\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2} \\
\frac{d^2y}{dx^2} &= \frac{-\frac{dy}{dx}\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2} \times \frac{\left(\frac{dy}{dx}\right)^2}{\left(\frac{dy}{dx}\right)^2} \\
\frac{d^2y}{dx^2} &= \frac{-\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3}{\left(\frac{dx}{dy}\frac{dy}{dx}\right)^2} \\
\frac{d^2y}{dx^2} &= -\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3
\end{aligned}$$

□

30. (a) Let θ be the angle between vectors **a** and **b**.

$$\begin{aligned}
|\mathbf{a}\cdot\mathbf{b}| &= |(|\mathbf{a}||\mathbf{b}|\cos\theta)| \\
&= |\mathbf{a}||\mathbf{b}||\cos\theta| \\
|\cos\theta| &\le 1 \\
\therefore \quad |\mathbf{a}\cdot\mathbf{b}| &\le |\mathbf{a}||\mathbf{b}|
\end{aligned}$$

□

(b)

$$\begin{aligned}
(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} \\
&= |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \\
|\mathbf{a}+\mathbf{b}|^2 &= |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \\
(|\mathbf{a}|+|\mathbf{b}|)^2 &= |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 \\
\therefore \quad |\mathbf{a}+\mathbf{b}|^2 &\le (|\mathbf{a}|+|\mathbf{b}|)^2 \\
\therefore \quad |\mathbf{a}+\mathbf{b}| &\le |\mathbf{a}|+|\mathbf{b}|
\end{aligned}$$

□

31. The radius of each ball is the height of its centre above the table, i.e. the **k** component.

The minimum possible distance between centres is twice the radius of the balls, i.e. 5cm. The cue ball will strike the 12 ball if the minimum distance between the centre of the 12 ball and the line along which the centre of the cue ball is travelling is less than 5cm.

Let A be the initial position of the cue ball, B be the position of the 12 ball and P be the point of closest approach between B and the line along which the cue ball is travelling.

$$\begin{aligned}
\overrightarrow{BP} &= \overrightarrow{BA} + \overrightarrow{AP} \\
&= (40-120)\mathbf{i} + (69-20)\mathbf{j} + (2.5-2.5)\mathbf{k} \\
&\quad + t\,(30\mathbf{i} - 20\mathbf{j}) \\
&= (-80\mathbf{i} + 49\mathbf{j}) + t\,(30\mathbf{i} - 20\mathbf{j})
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{BP}\cdot\overrightarrow{AP} &= 0 \\
((-80\mathbf{i}+49\mathbf{j}) + t\,(30\mathbf{i}-20\mathbf{j}))\cdot(30\mathbf{i}-20\mathbf{j}) &= 0 \\
(-80\mathbf{i}+49\mathbf{j})\cdot(30\mathbf{i}-20\mathbf{j}) & \\
+t\,(30\mathbf{i}-20\mathbf{j})\cdot(30\mathbf{i}-20\mathbf{j}) &= 0 \\
-2400 - 980 + t(900+400) &= 0 \\
1300t &= 3380 \\
t &= 2.6
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{BP} &= (-80\mathbf{i}+49\mathbf{j}) + 2.6\,(30\mathbf{i}-20\mathbf{j}) \\
&= (-2\mathbf{i} - 3\mathbf{j}) \\
|\overrightarrow{BP}| &= \sqrt{13} \\
&\approx 3.6
\end{aligned}$$

Therefore, the cue ball does strike the 12 ball.

32. Starting with the first expression stripped of the constant of integration:

$$\begin{aligned}
&-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x\\
&=\cos x\left(-1+\frac{2}{3}\cos^2 x-\frac{1}{5}\cos^4 x\right)\\
&=\cos x\left(-1+\frac{2(1-\sin^2 x)}{3}-\frac{(1-\sin^2 x)^2}{5}\right)\\
&=\cos x\left(-1+\frac{2}{3}-\frac{2\sin^2 x}{3}-\frac{1-2\sin^2 x+\sin^4 x}{5}\right)\\
&=\cos x\left(-1+\frac{2}{3}-\frac{1}{5}-\frac{2\sin^2 x}{3}+\frac{2\sin^2 x}{5}-\frac{\sin^4 x}{5}\right)\\
&=\cos x\left(\frac{-15+10-3}{15}+\frac{(-10+6)\sin^2 x}{15}-\frac{\sin^4 x}{5}\right)\\
&=\cos x\left(\frac{-8}{15}-\frac{4\sin^2 x}{15}-\frac{\sin^4 x}{5}\right)\\
&=\cos x\left(\frac{-\sin^4 x}{5}-\frac{4\sin^2 x}{15}-\frac{8}{15}\right)
\end{aligned}$$

which gives the first calculator display.

Now starting with some parts of the second calculator display:

$$\begin{aligned}
\cos 3x&=\cos(x+2x)\\
&=\cos x\cos 2x-\sin x\sin 2x\\
&=\cos x(2\cos^2 x-1)-\sin x(2\sin x\cos x)\\
&=2\cos^3 x-\cos x-2\sin^2 x\cos x\\
&=2\cos^3 x-\cos x-2(1-\cos^2 x)\cos x\\
&=2\cos^3 x-\cos x-2\cos x+2\cos^3 x\\
&=4\cos^3 x-3\cos x
\end{aligned}$$

$$\begin{aligned}
\cos 5x&=\cos(3x+2x)\\
&=\cos 3x\cos 2x-\sin 3x\sin 2x\\
&=(4\cos^3 x-3\cos x)(2\cos^2 x-1)-\sin(x+2x)\sin 2x\\
&=(8\cos^5 x-4\cos^3 x-6\cos^3 x+3\cos x)\\
&\quad-(\sin x\cos 2x+\cos x\sin 2x)(2\sin x\cos x)\\
&=(8\cos^5 x-10\cos^3 x+3\cos x)\\
&\quad-(\sin x(2\cos^2 x-1)+\cos x(2\sin x\cos x))(2\sin x\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-(2\sin x\cos^2 x-\sin x+2\sin x\cos^2 x)(2\sin x\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-(4\sin x\cos^2 x-\sin x)(2\sin x\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-(8\sin^2 x\cos^3 x-2\sin^2 x\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-\sin^2 x(8\cos^3 x-2\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-(1-\cos^2 x)(8\cos^3 x-2\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-(8\cos^3 x-2\cos x)+(8\cos^5 x-2\cos^3 x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-8\cos^3 x+2\cos x+8\cos^5 x-2\cos^3 x\\
&=16\cos^5 x-20\cos^3 x+5\cos x
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{-(150\cos x+3\cos 5x-25\cos 3x)}{240}\\
&\quad=\frac{-150\cos x}{240}\\
&\qquad-\frac{48\cos^5 x-60\cos^3 x+15\cos x}{240}\\
&\qquad-\frac{-100\cos^3 x+75\cos x}{240}\\
&\quad=\frac{-48\cos^5 x+160\cos^3 x-240\cos x)}{240}\\
&\quad=-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x
\end{aligned}$$

as required.