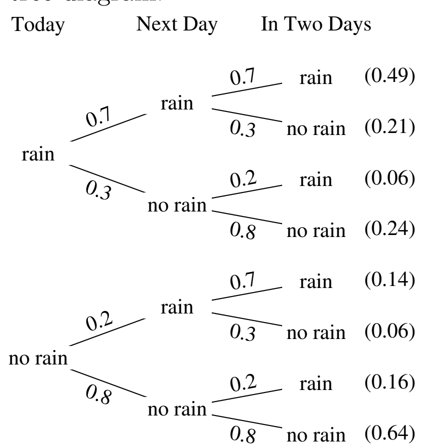
Chapter 5

Exercise 5A

1.

(a)
$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cc}\text{Rain today} & \text{No rain today}\end{array} \\ \text{To} \begin{array}{l}\text{Rain next day} \\ \text{No rain next day}\end{array} & \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \end{array}$$

(b) The tree diagram:



gives us the transition matrix
$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cc}\text{Rain today} & \text{No rain today}\end{array} \\ \text{To} \begin{array}{l}\text{Rain in 2 days} \\ \text{No rain in 2 days}\end{array} & \begin{bmatrix} 0.49+0.06 & 0.14+0.16 \\ 0.21+0.24 & 0.64+0.06 \end{bmatrix} \end{array}$$

which simplifies to
$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cc}\text{Rain today} & \text{No rain today}\end{array} \\ \text{To} \begin{array}{l}\text{Rain in 2 days} \\ \text{No rain in 2 days}\end{array} & \begin{bmatrix} 0.55 & 0.30 \\ 0.45 & 0.70 \end{bmatrix} \end{array}$$

and squaring the first matrix
$$\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}^2 = \begin{bmatrix} 0.55 & 0.30 \\ 0.45 & 0.70 \end{bmatrix}$$

gives the same result.

2. (a)
$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc}\text{Tom has ball 1 pass later} & \text{Tim has ball 1 pass later} & \text{Tony has ball 1 pass later}\end{array} \\ \text{From} \begin{array}{l}\text{Tom has ball now} \\ \text{Tim has ball now} \\ \text{Tony has ball now}\end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} \end{array}$$

(b) For two passes later, the probabilities are
$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}^2 = \begin{bmatrix} \frac{23}{40} & \frac{1}{8} & \frac{3}{10} \\ \frac{9}{20} & \frac{7}{20} & \frac{1}{5} \\ \frac{1}{10} & \frac{3}{8} & \frac{21}{40} \end{bmatrix}$$

For three passes later, the probabilities are
$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}^3 = \begin{bmatrix} \frac{11}{40} & \frac{29}{80} & \frac{29}{80} \\ \frac{29}{100} & \frac{11}{40} & \frac{87}{200} \\ \frac{87}{160} & \frac{29}{160} & \frac{11}{40} \end{bmatrix}$$

Using these,

i. From Tom to Tony after two passes, refer to the cell in the first row (from Tom) and third column (to Tony): the probability is $\frac{3}{10}$.

ii. From Tony to Tim after two passes, refer to the cell in the third row (from Tony) and second column (to Tim): the probability is $\frac{3}{8}$.

iii. From Tim back to Tim after three passes, refer to the cell in the second row (from Tim) and second column (to Tim): the probability is $\frac{11}{40}$.

iv. From Tom back to Tom after three passes, refer to the cell in the first row (from Tom) and first column (to Tom): the probability that he will have the ball is $\frac{11}{40}$, so the probability that he will not have the ball is $\frac{29}{40}$.

3. Let R be the transition matrix for Roz's coffee shop visits:
$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc}\text{A next week} & \text{B next week} & \text{C next week}\end{array} \\ \text{From} \begin{array}{l}\text{A this week} \\ \text{B this week} \\ \text{C this week}\end{array} & \begin{bmatrix} \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \end{array}$$

(a) From cell (3,3) of R, $p = \frac{1}{2}$.

(b) $R^3 = \begin{bmatrix} 0.228 & 0.300 & 0.472 \\ 0.245 & 0.277 & 0.479 \\ 0.239 & 0.284 & 0.477 \end{bmatrix}$
so $p = 0.477$ (3 d.p.).

(c) $R^{10} = \begin{bmatrix} 0.238 & 0.286 & 0.476 \\ 0.238 & 0.286 & 0.476 \\ 0.238 & 0.286 & 0.476 \end{bmatrix}$
so $p = 0.476$ (3 d.p.).
Notice how all three rows are identical to three decimal places. This indicates that after ten weeks it makes no difference to the probabilities (at this level of precision) which coffee shop Roz started in. We have reached the long-range expectation for the three coffee shops.

4. The transition matrix T is
$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc}\text{Labour next election} & \text{Conservative next election} & \text{Other next election}\end{array} \\ \text{From} \begin{array}{l}\text{Labour this election} \\ \text{Conservative this election} \\ \text{Other this election}\end{array} & \begin{bmatrix} 0.81 & 0.07 & 0.12 \\ 0.13 & 0.78 & 0.09 \\ 0.13 & 0.12 & 0.75 \end{bmatrix} \end{array}$$

(a) $T^3 = \begin{bmatrix} 0.59 & 0.17 & 0.24 \\ 0.28 & 0.52 & 0.20 \\ 0.28 & 0.24 & 0.49 \end{bmatrix}$

so the probability of a Labour voter in one election voting Labour three elections later is 0.59 (from cell (1,1)).

(b) To vote Labour in *each* of the next three elections, the probability is $0.81^3 = 0.53$.

(c) In addition to consistently voting Labour, answer (a) includes all combinations of voting behaviour that start with Labour, have votes for another party in the first and/or second subsequent elections and then return to Labour for the third election.

(d) $T^4 = \begin{bmatrix} 0.53 & 0.20 & 0.26 \\ 0.32 & 0.45 & 0.23 \\ 0.32 & 0.26 & 0.42 \end{bmatrix}$

The probability that a voter who votes Conservative in an election votes Other in four elections time is 0.23 (from cell (2,3)).

5. It's possible to answer (a) directly from the transition diagram (the probability of a transition from B to A is 0.2) and (b) and perhaps (c) could be answered using a tree diagram, but by far the simplest approach is to translate the transition diagram to a transition matrix T:

$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \text{From} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{bmatrix} 0.1 & 0.6 & 0.3 \\ 0.2 & 0.4 & 0.4 \\ 0.8 & 0 & 0.2 \end{bmatrix} \end{array}$$

(The transition matrix can also be written with the columns representing the initial state and the rows representing the resulting state.)

The probability of the situation being in state A given that it starts in state B is found in cell $(2, 1)$

(a) $p = 0.2$

(b) $T^2 = \begin{bmatrix} 0.37 & 0.30 & 0.33 \\ 0.42 & 0.28 & 0.30 \\ 0.24 & 0.48 & 0.28 \end{bmatrix}$

$p = 0.42$

(c) $T^3 = \begin{bmatrix} 0.361 & 0.342 & 0.297 \\ 0.338 & 0.364 & 0.298 \\ 0.344 & 0.336 & 0.320 \end{bmatrix}$

$p = 0.338$

(d) $T^5 = \begin{bmatrix} 0.348 & 0.347 & 0.305 \\ 0.349 & 0.346 & 0.304 \\ 0.345 & 0.351 & 0.304 \end{bmatrix}$

$p = 0.349$

(e) $T^{10} = \begin{bmatrix} 0.348 & 0.348 & 0.304 \\ 0.348 & 0.348 & 0.304 \\ 0.348 & 0.348 & 0.304 \end{bmatrix}$

$p = 0.348$

6. The transition diagram translates to the following transition matrix T:

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} \text{P} & \text{Q} & \text{R} \end{array} \\ \text{To} \begin{array}{c} \text{P} \\ \text{Q} \\ \text{R} \end{array} & \begin{bmatrix} 0.2 & 0.4 & 0.1 \\ 0.8 & 0 & 0.9 \\ 0 & 0.6 & 0 \end{bmatrix} \end{array}$$

After starting in state R, the probability of being in state P is found in cell $(1, 3)$, the probability of being in state Q is in cell $(2, 3)$ and the probability of being in state R is in $(3, 3)$.

(a) $p(\text{P}) = 0.1$

(b) $T^2 = \begin{bmatrix} 0.36 & 0.14 & 0.38 \\ 0.16 & 0.86 & 0.08 \\ 0.48 & 0 & 0.54 \end{bmatrix}$

$p(\text{Q}) = 0.08$

(c) $T^3 = \begin{bmatrix} 0.184 & 0.372 & 0.162 \\ 0.720 & 0.112 & 0.790 \\ 0.096 & 0.516 & 0.048 \end{bmatrix}$

$p(\text{R}) = 0.048$

(d) $T^{10} = \begin{bmatrix} 0.295 & 0.224 & 0.303 \\ 0.362 & 0.590 & 0.336 \\ 0.343 & 0.186 & 0.361 \end{bmatrix}$

$p(\text{P}) = 0.303$

(e) $T^{20} = \begin{bmatrix} 0.272 & 0.254 & 0.274 \\ 0.436 & 0.492 & 0.429 \\ 0.292 & 0.254 & 0.297 \end{bmatrix}$

$p(\text{Q}) = 0.429$

Exercise 5B

1. Transition matrix T:

$$\begin{array}{ccc} & & \text{From} \\ & & \begin{array}{cc}\text{City now} & \text{Country now}\end{array} \\ \text{To} & \begin{array}{l}\text{City in 5 years} \\ \text{Country in 5 years}\end{array} & \begin{bmatrix} 0.95 & 0.18 \\ 0.05 & 0.82 \end{bmatrix} \end{array}$$

(a) i. $\begin{bmatrix} 0.95 & 0.18 \\ 0.05 & 0.82 \end{bmatrix}\begin{bmatrix} 765 \\ 511 \end{bmatrix} = \begin{bmatrix} 819 \\ 457 \end{bmatrix}$
The model predicts 819 000 will live in the city and 457 000 in the country in five years time.

ii. $\begin{bmatrix} 0.95 & 0.18 \\ 0.05 & 0.82 \end{bmatrix}^2\begin{bmatrix} 765 \\ 511 \end{bmatrix} = \begin{bmatrix} 860 \\ 416 \end{bmatrix}$
The model predicts 860 000 will live in the city and 416 000 in the country in five years time.

(b) Experiment with increasingly high powers of T until both columns are equal, to two decimals:

$$T^{20} = \begin{bmatrix} 0.78 & 0.78 \\ 0.22 & 0.22 \end{bmatrix}$$

In the long term 78 per cent of the population will live in the city and 22 per cent in the country.

(c) I postmultiplied the transition matrix by the initial state matrix because of the way I set up my transition matrix with the columns representing the 'From' state and the rows representing the 'To' state.

2. Transition matrix T:

$$\begin{array}{ccc} & & \text{To} \\ & & \begin{array}{cc}\text{A} & \text{B}\end{array} \\ \text{From} & \begin{array}{c}\text{A} \\ \text{B}\end{array} & \begin{bmatrix} 0.98 & 0.02 \\ 0.05 & 0.95 \end{bmatrix} \end{array}$$

(a) $\begin{bmatrix} 260 & 138 \end{bmatrix}\begin{bmatrix} 0.98 & 0.02 \\ 0.05 & 0.95 \end{bmatrix}^2 = \begin{bmatrix} 263 & 135 \end{bmatrix}$
The model predicts that 263 of the 398 staff, or 66% will be at A in two years time, with 34% at B.

(b) $\begin{bmatrix} 260 & 138 \end{bmatrix}\begin{bmatrix} 0.98 & 0.02 \\ 0.05 & 0.95 \end{bmatrix}^5 = \begin{bmatrix} 267 & 131 \end{bmatrix}$
The model predicts that 267 of the 398 staff, or 67% will be at A in five years time, with 33% at B.

(c) Experiment with increasingly high powers of T until both rows are equal, to two decimals:

$$T^{100} = \begin{bmatrix} 0.71 & 0.29 \\ 0.71 & 0.29 \end{bmatrix}$$

and conclude that the long term expectation is that 71% will be at A and 29% at B.
Alternatively, if an exact value is required solve the steady state equation

$$\begin{bmatrix} a & 1-a \end{bmatrix}\begin{bmatrix} 0.98 & 0.02 \\ 0.05 & 0.95 \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix}$$

$$\begin{bmatrix} 0.98a + 0.05(1-a) & 0.02a + 0.95(1-a) \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix}$$

equating the first elements,

$$\begin{aligned} 0.98a + 0.05(1-a) &= a \\ 0.05 - 0.05a &= 0.02a \\ 0.07a &= 0.05 \\ a &= \frac{5}{7} \\ &\approx 0.71 \end{aligned}$$

similarly the second elements

$$\begin{aligned} 0.02a + 0.95(1-a) &= 1-a \\ 0.02a &= 0.05(1-a) \\ 0.07a &= 0.05 \\ a &= \frac{5}{7} \\ &\approx 0.71 \end{aligned}$$

3. (a) Cell $(1,1)$: $p = \frac{1}{2}$.

(b) $\begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}^2 = \begin{bmatrix} \frac{121}{300} & \frac{13}{50} & \frac{101}{300} \\ \frac{3}{10} & \frac{7}{20} & \frac{7}{20} \\ \frac{59}{150} & \frac{13}{50} & \frac{26}{75} \end{bmatrix}$
Cell $(2,3)$: $p = \frac{7}{20}$

(c) To two decimal places,

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}^5 = \begin{bmatrix} 0.37 & 0.29 & 0.34 \\ 0.37 & 0.29 & 0.34 \\ 0.37 & 0.29 & 0.34 \end{bmatrix}$$

In the long term, the team expects to win 37%, lose 29% and draw 34%.

4. First present the information as initial matrix I and transition matrix T:

$$I = \begin{array}{l} \text{Tea} \\ \text{Coffee} \\ \text{Juice} \\ \text{No drink} \end{array}\begin{bmatrix} 52 \\ 93 \\ 84 \\ 21 \end{bmatrix}$$

$$T = \begin{array}{ccc} & & \text{Today} \\ & & \begin{array}{cccc}\text{Tea} & \text{Coffee} & \text{Juice} & \text{No drink}\end{array} \\ \begin{array}{c}\text{Next} \\ \text{day}\end{array} & \begin{array}{l}\text{Tea} \\ \text{Coffee} \\ \text{Juice} \\ \text{No drink}\end{array} & \begin{bmatrix} 0.65 & 0.08 & 0.04 & 0.16 \\ 0.22 & 0.75 & 0.08 & 0.24 \\ 0.09 & 0.15 & 0.82 & 0.25 \\ 0.04 & 0.02 & 0.06 & 0.35 \end{bmatrix} \end{array}$$

(a) $TI = \begin{bmatrix} 48 \\ 93 \\ 93 \\ 16 \end{bmatrix}$
The probabilities suggest 48 tea, 93 coffee, 93 juice and 16 no drink the next day.

(b) $T^{30} = \begin{bmatrix} 0.16 & 0.16 & 0.16 & 0.16 \\ 0.34 & 0.34 & 0.34 & 0.34 \\ 0.44 & 0.44 & 0.44 & 0.44 \\ 0.06 & 0.06 & 0.06 & 0.06 \end{bmatrix}$

The probabilities suggest the long term percentages are 16% tea, 34% coffee, 44% juice and 6% no drink.

5. Since the question does not spell out exactly what "randomly select" means, we must assume equal probability for each available path. A person at A or D has three available paths so we assign each the probability of $\frac{1}{3}$ while a person at B or C has only two available paths so we assign each of these the probability of $\frac{1}{2}$. This gives the transition matrix

$$T = \begin{array}{cc} & \text{To} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{From} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 22 & 22 & 22 & 22 \end{bmatrix} T^{10} = \begin{bmatrix} 27 & 16 & 24 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 88 & 0 & 0 & 0 \end{bmatrix} T^{10} = \begin{bmatrix} 27 & 16 & 24 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 44 & 0 & 44 \end{bmatrix} T^{10} = \begin{bmatrix} 27 & 16 & 24 & 21 \end{bmatrix}$$

etc.

In the long term, after (say) ten moves, it makes no difference where the 88 people are initially stationed, they will end up with 27 on A, 16 on B, 24 on C and 21 on D.

(These are, of course, only expected outcomes. Because of the random nature of the moves the actual numbers would vary either side of these expected values.)

6. First complete the transition matrix so each column totals 1.0:

$$\begin{bmatrix} 0.61 & 0.28 & 0.19 \\ 0.35 & 0.64 & 0.72 \\ 0.04 & 0.08 & 0.09 \end{bmatrix}$$

(a) $$\begin{bmatrix} 0.61 & 0.28 & 0.19 \\ 0.35 & 0.64 & 0.72 \\ 0.04 & 0.08 & 0.09 \end{bmatrix} \begin{bmatrix} 35 \\ 48 \\ 12 \end{bmatrix} = \begin{bmatrix} 37 \\ 52 \\ 6 \end{bmatrix}$$

The table suggests that 37 will have school, 52 a normal degree and 6 a higher degree as their highest level of education.

(b) $$\begin{bmatrix} 0.61 & 0.28 & 0.19 \\ 0.35 & 0.64 & 0.72 \\ 0.04 & 0.08 & 0.09 \end{bmatrix}^6 = \begin{bmatrix} 0.41 & 0.41 & 0.41 \\ 0.53 & 0.53 & 0.53 \\ 0.06 & 0.06 & 0.06 \end{bmatrix}$$

If the trends continue, in the long term 41% will have school, 53% a normal degree and 6% a higher degree as their highest level of education. (However, given the nature of the data, it might be considered very unlikely for these trends to continue unchanged for the several generations needed for the long-term outcome to be meaningful. Also note other potential issues with the experimental design mentioned in Sadler's answers.)

7. Completing the matrix gives

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{To} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \end{array}$$

For the long-term probabilities,

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix}^{15} = \begin{bmatrix} 0.286 & 0.286 & 0.286 \\ 0.429 & 0.429 & 0.429 \\ 0.286 & 0.286 & 0.286 \end{bmatrix}$$

$p(A) = 0.286$; $p(B) = 0.429$; and $p(C) = 0.286$. Intuitively we would expect the probability of finding the sentry at B to be greater than that of finding him at A or C, and we would have expected to find the symmetry with A and C having equal probability, but we would not have predicted the actual probabilities intuitively.

8. The transition matrix is

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} X & Y & Z \end{array} \\ \text{To} \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{2}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{3} & 0 & 0 \end{bmatrix} \end{array}$$

For the long-term probabilities,

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{2}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{3} & 0 & 0 \end{bmatrix}^{15} = \begin{bmatrix} 0.474 & 0.474 & 0.474 \\ 0.211 & 0.211 & 0.211 \\ 0.316 & 0.316 & 0.316 \end{bmatrix}$$

$p(X) = 0.474$; $p(Y) = 0.211$; and $p(Z) = 0.316$.

(These probabilities correct to three decimal places appear to add to 1.001. This is, of course, simply an artefact of the rounding and we should not be disconcerted by it.)

Exercise 5C

1. (a) $L^2P = \begin{bmatrix} 3\,888 \\ 3\,318 \\ 1\,792 \\ 864 \end{bmatrix}$

$L^5P = \begin{bmatrix} 6\,179 \\ 4\,130 \\ 2\,977 \\ 1\,306 \end{bmatrix}$

$L^{10}P = \begin{bmatrix} 11\,807 \\ 7\,477 \\ 5\,196 \\ 2\,751 \end{bmatrix}$

$L^{20}P = \begin{bmatrix} 40\,108 \\ 24\,867 \\ 17\,607 \\ 9\,370 \end{bmatrix}$

$L^{50}P = \begin{bmatrix} 1\,530\,091 \\ 948\,631 \\ 672\,154 \\ 357\,192 \end{bmatrix}$

(b) The product is a 1×1 matrix where the cell value is the total population.

(c)
$$\begin{aligned}
TL^5P &= \begin{bmatrix} 14\,593 \end{bmatrix} & x_5 &= 14\,593\\
TL^6P &= \begin{bmatrix} 17\,003 \end{bmatrix} & x_6 &= 17\,003\\
TL^7P &= \begin{bmatrix} 18\,942 \end{bmatrix} & x_7 &= 18\,942\\
TL^8P &= \begin{bmatrix} 21\,335 \end{bmatrix} & x_8 &= 21\,335\\
TL^9P &= \begin{bmatrix} 24\,312 \end{bmatrix} & x_9 &= 24\,312\\
TL^{10}P &= \begin{bmatrix} 27\,231 \end{bmatrix} & x_{10} &= 27\,231\\
TL^{19}P &= \begin{bmatrix} 81\,436 \end{bmatrix} & x_{19} &= 81\,436\\
TL^{20}P &= \begin{bmatrix} 91\,952 \end{bmatrix} & x_{20} &= 91\,952\\
TL^{29}P &= \begin{bmatrix} 274\,149 \end{bmatrix} & x_{29} &= 274\,149\\
TL^{30}P &= \begin{bmatrix} 309\,534 \end{bmatrix} & x_{30} &= 309\,534\\
TL^{39}P &= \begin{bmatrix} 922\,931 \end{bmatrix} & x_{39} &= 922\,931\\
TL^{40}P &= \begin{bmatrix} 1\,042\,047 \end{bmatrix} & x_{40} &= 1\,042\,047\\
TL^{49}P &= \begin{bmatrix} 3\,107\,062 \end{bmatrix} & x_{49} &= 3\,107\,062\\
TL^{50}P &= \begin{bmatrix} 3\,508\,068 \end{bmatrix} & x_{50} &= 3\,508\,068
\end{aligned}$$

(d)
$$\begin{aligned}
\frac{x_6}{x_5} &= 1.165\\
\frac{x_7}{x_6} &= 1.114\\
\frac{x_8}{x_7} &= 1.126\\
\frac{x_9}{x_8} &= 1.140\\
\frac{x_{10}}{x_9} &= 1.120\\
\frac{x_{20}}{x_{19}} &= 1.129\\
\frac{x_{30}}{x_{29}} &= 1.129\\
\frac{x_{40}}{x_{39}} &= 1.129\\
\frac{x_{50}}{x_{49}} &= 1.129
\end{aligned}$$

These results suggest a long-term steady growth rate of 12.9% per generation.

2. (a) The 0.3 is multiplied by the Youngster population and the product contributes to the Infant population in the next generation: that is, it is the reproduction rate of Youngsters.

(b) The 0.9 is multiplied by the Prime population and the product gives the Elderly population in the next generation: that is, it is the survival rate of the Prime population.

(c) $P_1 = \begin{bmatrix} 0 & 0.3 & 1.8 & 0.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 250 \\ 540 \\ 620 \\ 280 \\ 140 \end{bmatrix} = \begin{bmatrix} 1420 \\ 175 \\ 430 \\ 560 \\ 140 \end{bmatrix}$

Translate into the terms of the question:

Gen.	Infant	Youngster	Prime	Elderly	Aged
Pop'n	1420	175	430	560	140

(Note: these are rounded figures.)

(d) $P_5 = \begin{bmatrix} 0 & 0.3 & 1.8 & 0.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}^5 \begin{bmatrix} 250 \\ 540 \\ 620 \\ 280 \\ 140 \end{bmatrix} = \begin{bmatrix} 1630 \\ 1210 \\ 420 \\ 560 \\ 360 \end{bmatrix}$

Gen.	Infant	Youngster	Prime	Elderly	Aged
Pop'n	1630	1210	420	560	360

(e) $$\mathrm{P}_{25} = \begin{bmatrix} 0 & 0.3 & 1.8 & 0.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}^{25} \begin{bmatrix} 250 \\ 540 \\ 620 \\ 280 \\ 140 \end{bmatrix} = \begin{bmatrix} 20\,100 \\ 12\,300 \\ 8\,800 \\ 6\,900 \\ 3\,000 \end{bmatrix}$$

Gen.	Infant	Youngster	Prime	Elderly	Aged
Pop'n	20 100	12 300	8 800	6 900	3 000

(f) $$\begin{bmatrix} 0 & 0.3 & 1.8 & 0.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}^{10} \begin{bmatrix} 250 \\ 540 \\ 620 \\ 280 \\ 140 \end{bmatrix} = \begin{bmatrix} 3\,100 \\ 1\,600 \\ 1\,300 \\ 1\,100 \\ 350 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3\,100 \\ 1\,600 \\ 1\,300 \\ 1\,100 \\ 350 \end{bmatrix} = \begin{bmatrix} 7600 \end{bmatrix}$$

The total population after 20 years (10 generations) is 7 600 (to the nearest hundred).

3. (a) No working required. Put reproduction rates in the first row and survival rates (not death rates!) in the first diagonal below the prime diagonal.

(b) $$\begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^{10} \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 308\,000 \\ 143\,000 \\ 76\,000 \\ 21\,000 \end{bmatrix}$$

(rounded to the nearest thousand).

(c) $$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^{25} \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 243\,000\,000 \end{bmatrix}$$

(rounded to the nearest million).

(d) From generation 25 to generation 26 the total population increases by

$$\frac{364961828}{243089970} = 1.5013$$

From generation 26 to generation 27 the total population increases by

$$\frac{364961828}{243089970} = 1.5013$$

Since these growth factors are equal to four decimal places, we can assume that we have reached the long-term growth rate of 50% every two years.

(e) I would expect the investigation to proceed somewhat along these lines:
Initial population:

$$\mathrm{P}_0 = \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = 101 \begin{bmatrix} 0.23 \\ 0.28 \\ 0.32 \\ 0.18 \end{bmatrix}$$

Subsequent generations:

$$\mathrm{P}_1 = \begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 12060 \\ 1610 \\ 2240 \\ 1280 \end{bmatrix} = 17190 \begin{bmatrix} 0.70 \\ 0.09 \\ 0.13 \\ 0.07 \end{bmatrix}$$

$$\mathrm{P}_2 = \begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^2 \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 7639 \\ 8442 \\ 1288 \\ 896 \end{bmatrix} = 18256 \begin{bmatrix} 0.42 \\ 0.46 \\ 0.07 \\ 0.05 \end{bmatrix}$$

$$\mathrm{P}_3 = \begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^3 \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 21696 \\ 5347 \\ 6754 \\ 515 \end{bmatrix} = 34312 \begin{bmatrix} 0.63 \\ 0.16 \\ 0.20 \\ 0.02 \end{bmatrix}$$

$$\mathrm{P}_4 = \begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^4 \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 23831 \\ 15187 \\ 4278 \\ 2701 \end{bmatrix} = 45997 \begin{bmatrix} 0.52 \\ 0.33 \\ 0.09 \\ 0.06 \end{bmatrix}$$

Proceeding in this manner and tabulating the proportions in each generation gives

Gen	$0 < x < 2$	$2 \le x < 4$	$4 \le x < 6$	$6 \le x < 8$
0	0.23	0.28	0.32	0.18
1	0.70	0.09	0.13	0.07
2	0.42	0.46	0.07	0.05
3	0.63	0.16	0.20	0.02
4	0.52	0.33	0.09	0.06
5	0.58	0.23	0.17	0.02
6	0.55	0.28	0.12	0.05
7	0.56	0.26	0.15	0.03
8	0.56	0.26	0.14	0.04
9	0.56	0.26	0.14	0.04
10	0.56	0.26	0.14	0.04
11	0.56	0.26	0.14	0.04
12	0.56	0.26	0.14	0.04

At first the proportions vary wildly, but after a few generations they begin to settle down and after 8 generations there is no change (at least at this level of precision) from 56% in the youngest age group, 26% in the second, 14% in the third and 4% in the oldest.

Exercise 5D

1. (a)

$$L = \begin{bmatrix} 0 & 0 & 0.8 & 0.4 & 0.1 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 350 \\ 420 \\ 330 \\ 140 \\ 70 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} TP_1 &= TLP_0 \\ &= \begin{bmatrix} 989 \end{bmatrix} \\ TP_2 &= TL^2P_0 \\ &= \begin{bmatrix} 770 \end{bmatrix} \\ TP_2 &= TL^3P_0 \\ &= \begin{bmatrix} 517 \end{bmatrix} \\ TP_2 &= TL^4P_0 \\ &= \begin{bmatrix} 375 \end{bmatrix} \\ TP_2 &= TL^5P_0 \\ &= \begin{bmatrix} 289 \end{bmatrix} \\ TP_2 &= TL^{10}P_0 \\ &= \begin{bmatrix} 55 \end{bmatrix} \end{aligned}$$

The total population in 1, 2, 3, 4, 5 and 10 generations time is predicted by the model to be 989, 770, 517, 375, 289 and 55.

(b)

$$L = \begin{bmatrix} 0 & 0 & 0.8 & 0.4 & 0.1 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}$$

$$\begin{aligned} TP_1 &= TLP_0 \\ &= \begin{bmatrix} 1319 \end{bmatrix} \\ TP_2 &= TL^2P_0 \\ &= \begin{bmatrix} 1363 \end{bmatrix} \\ TP_2 &= TL^3P_0 \\ &= \begin{bmatrix} 1256 \end{bmatrix} \\ TP_2 &= TL^4P_0 \\ &= \begin{bmatrix} 1141 \end{bmatrix} \\ TP_2 &= TL^5P_0 \\ &= \begin{bmatrix} 1127 \end{bmatrix} \\ TP_2 &= TL^{10}P_0 \\ &= \begin{bmatrix} 848 \end{bmatrix} \end{aligned}$$

The total population in 1, 2, 3, 4, 5 and 10 generations time is predicted by the revised model to be 1319, 1363, 1256, 1141, 1127 and 848.

(c) $$L = \begin{bmatrix} 0 & 0.2 & 0.9 & 0.5 & 0.1 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}$$

$$\begin{aligned} TP_1 &= TLP_0 \\ &= \begin{bmatrix} 1450 \end{bmatrix} \\ TP_2 &= TL^2P_0 \\ &= \begin{bmatrix} 1588 \end{bmatrix} \\ TP_2 &= TL^3P_0 \\ &= \begin{bmatrix} 1575 \end{bmatrix} \\ TP_2 &= TL^4P_0 \\ &= \begin{bmatrix} 1620 \end{bmatrix} \\ TP_2 &= TL^5P_0 \\ &= \begin{bmatrix} 1736 \end{bmatrix} \\ TP_2 &= TL^{10}P_0 \\ &= \begin{bmatrix} 2054 \end{bmatrix} \end{aligned}$$

The total population in 1, 2, 3, 4, 5 and 10 generations time is predicted by the second revised model to be 1450, 1588, 1575, 1620, 1736 and 2054.

2. (a) No working required.

(b) No working required.

(c) i. $$\begin{bmatrix} 0 & 0.5 & 0.9 & 1.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 350 \\ 620 \\ 750 \\ 180 \\ 60 \end{bmatrix} = \begin{bmatrix} 1255 \\ 245 \\ 496 \\ 675 \\ 108 \end{bmatrix}$$

ii. $$\begin{bmatrix} 0 & 0.5 & 0.9 & 1.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \end{bmatrix}^{10} \begin{bmatrix} 350 \\ 620 \\ 750 \\ 180 \\ 60 \end{bmatrix} = \begin{bmatrix} 4082 \\ 2597 \\ 1695 \\ 1246 \\ 711 \end{bmatrix}$$

(d) Let the current population figures be represented by

$$P_0 = \begin{bmatrix} 350 \\ 620 \\ 750 \\ 180 \\ 60 \end{bmatrix}$$

and let U be given by

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

then the total population after n years is given by

$$P_n = UL^nP_0$$

i. $UL^{19}P_0 = \begin{bmatrix} 39\,380 \end{bmatrix}$

ii. $UL^{20}P_0 = \begin{bmatrix} 45\,714 \end{bmatrix}$

iii. $UL^{29}P_0 = \begin{bmatrix} 174\,005 \end{bmatrix}$

iv. $UL^{30}P_0 = \begin{bmatrix} 201\,888 \end{bmatrix}$

(e) $\dfrac{45\,714}{39\,380} \approx 1.16$ The long term annual per-
$\dfrac{201\,888}{174\,005} \approx 1.16$
centage growth rate is about 16%.

(f) $\frac{1}{1.16} = 0.86$ so the annual harvesting rate should be about 14%.

(g) $$(0.95L)^5P_0 = \begin{bmatrix} 1699 \\ 786 \\ 557 \\ 617 \\ 294 \end{bmatrix} \approx \begin{bmatrix} 1700 \\ 800 \\ 550 \\ 600 \\ 300 \end{bmatrix}$$

3. (a) $$L = \begin{bmatrix} 0 & 0.8 & 1.6 & 0.3 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \end{bmatrix}$$

(b) Let

$$P_0 = \begin{bmatrix} 850 \\ 750 \\ 600 \\ 400 \end{bmatrix}$$

and let

$$I = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

then

i. $IL^9P_0 = \begin{bmatrix} 7250 \end{bmatrix}$

ii. $IL^{10}P_0 = \begin{bmatrix} 8134 \end{bmatrix}$

iii. $IL^{19}P_0 = \begin{bmatrix} 21968 \end{bmatrix}$

iv. $IL^{20}P_0 = \begin{bmatrix} 24542 \end{bmatrix}$

v. $IL^{29}P_0 = \begin{bmatrix} 66558 \end{bmatrix}$

vi. $IL^{30}P_0 = \begin{bmatrix} 74359 \end{bmatrix}$

(c) $\dfrac{P_{10}}{P_9} = 1.122$

$\dfrac{P_{20}}{P_{19}} = 1.119$

$\dfrac{P_{30}}{P_{29}} = 1.117$

The data suggests the long term annual growth rate will be about 12%.

(d) $\dfrac{1}{1.117} = 0.895$

$1 - 0.895 = 0.105$

The annual harvesting rate should be between 10% and 11%.

To estimate the long term steady population of each year group, consider P_{30} with 10.5% harvesting:

$$P_{30} = (0.895L)^{30}\,P_0 = \begin{bmatrix} 1230 \\ 660 \\ 470 \\ 300 \end{bmatrix}$$

(answers rounded to the nearest 10.)

(e) $\dfrac{0.95}{1.117} = 0.850$

$1 - 0.850 = 0.15$

The annual harvesting rate should be about 15%.

$$P_{10} = (0.850L)^{10}\,P_0 = \begin{bmatrix} 740 \\ 400 \\ 280 \\ 180 \end{bmatrix}$$

(answers rounded to the nearest 10.)

Miscellaneous Exercise 5

1. (a) AB cannot be determined because it would require the number of columns of A (1) to equal the number of rows of B (2).

   (b) $$\text{BA} = \begin{bmatrix} -1\times 3+2\times 1 \\ 1\times 3+4\times 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

   (c) $$\text{BC} = \begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \end{bmatrix}$$

   (d) CD cannot be determined because it would require the number of columns of C (3) to equal the number of rows of D (2).

   (e) $$\text{BD} = \begin{bmatrix} 0 & 1 & 2 \\ 6 & 5 & 4 \end{bmatrix}$$

2. (a) $$\text{AB} = \begin{bmatrix} 1\times 2-2\times 0+2\times -1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

   (b) $$\text{BA} = \begin{bmatrix} 2\times 1 & 2\times -2 & 2\times 2 \\ 0\times 1 & 0\times -2 & 0\times 2 \\ -1\times 1 & -1\times -2 & -1\times 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 4 \\ 0 & 0 & 0 \\ -1 & 2 & -2 \end{bmatrix}$$

3. $$\begin{aligned} \text{AC} &= \text{B} \\ \text{A}^{-1}\text{AC} &= \text{A}^{-1}\text{B} \\ \text{IC} &= \text{A}^{-1}\text{B} \\ \text{C} &= \text{A}^{-1}\text{B} \\ &= \frac{1}{2\times 4-2\times -1}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 21 \\ 9 & 17 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 16-27 & 84-51 \\ 4+18 & 21+34 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} -11 & 33 \\ 22 & 55 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix} \end{aligned}$$

4. (a) XY (2×2)(2×1) and ZX (1×2)(2×2) can be formed since these have the correct match of the number of columns in the first matrix with the number of rows in the second.

   (b) XY does not make sense. Consider the calculation of the value in the first cell: (No. of Aus stamps in the Mainly Aus pack × No. of Mainly Aus packs) + (No. of World stamps in the Mainly Aus pack × No. of Rest of World packs). Although the first term makes sense (giving the total number of Aus stamps in the Mainly Aus pack), the second term makes no sense since it multiplies two unrelated quantities.

   ZX does make sense. Consider again the calculation of the value in the first cell: (No. of Aus stamps in the Mainly Aus pack × No. of Mainly Aus packs) + (No. of Aus stamps in the Rest of World pack × No. of Rest of World packs). The two terms here give the total number of Aus stamps in the Mainly Aus pack and the total number of Aus stamps in the Rest of World pack, and their sum gives the total number of Aus stamps required. Similarly the value in the second cell gives the total number of Rest of World stamps required.

   (c) $$\text{ZX} = \begin{bmatrix} 210 & 120 \end{bmatrix}\begin{bmatrix} 75 & 25 \\ 20 & 80 \end{bmatrix} = \begin{bmatrix} 18150 & 14850 \end{bmatrix}$$

   18150 Australian stamps and 14850 Rest of World stamps will be required in order to supply the requests.

5. From matrix element (1,1):

$$\begin{aligned} 45 - x^2 &= 4x \\ x^2 + 4x - 45 &= 0 \end{aligned}$$

From matrix element (2,2):

$$\begin{aligned} 6x - 5 &= x^2 \\ x^2 - 6x + 5 &= 0 \end{aligned}$$

Combining these:

$$\begin{aligned} x^2 + 4x - 45 &= x^2 - 6x + 5 \\ 10x &= 50 \\ x &= 5 \end{aligned}$$

Similarly, from matrix element (1,2):

$$\begin{aligned} y^2 - y &= 4 - y \\ y^2 - 4 &= 0 \end{aligned}$$

From matrix element (2,1):

$$\begin{aligned} y^2 + 5y &= -6 \\ y^2 - 5y + 6 &= 0 \end{aligned}$$

Combining these:

$$\begin{aligned} y^2 - 4 &= y^2 - 5y + 6 \\ -5y &= -10 \\ y &= 2 \end{aligned}$$

6.
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6x(x-2)}{2y-3}\\
\int (2y-3)dy &= \int (6x(x-2))\,dx\\
&= \int \left(6x^2-12x\right)dx\\
\text{at } (3,4){:}\quad y^2-3y &= 2x^3-6x^2+c\\
(4)^2-3(4) &= 2(3)^3-6(3)^2+c\\
16-12 &= 54-54+c\\
c &= 4\\
y^2-3y &= 2x^3-6x^2+4\\
\text{given } x=1,\quad y^2-3y &= 2(1)^3-6(1)^2+4\\
&= 2-6+4\\
&= 0\\
y(y-3) &= 0\\
y &= 0\\
\text{or}\quad y &= 3
\end{aligned}$$

7. (a)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} \cos(90^\circ) & -\sin(90^\circ)\\ \sin(90^\circ) & \cos(90^\circ)\end{bmatrix}\\
&= \begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}\\
|\det(\mathrm{T})| &= |0\times 0-(-1)\times 1|\\
&= 1
\end{aligned}$$

(b)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} \cos(180^\circ) & -\sin(180^\circ)\\ \sin(180^\circ) & \cos(180^\circ)\end{bmatrix}\\
&= \begin{bmatrix} -1 & 0\\ 0 & -1\end{bmatrix}\\
|\det(\mathrm{T})| &= |-1\times -1-0\times 0|\\
&= 1
\end{aligned}$$

(c)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}\\
|\det(\mathrm{T})| &= |1\times -1-0\times 0|\\
&= 1
\end{aligned}$$

(d)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}\\
|\det(\mathrm{T})| &= |0\times 0-1\times 1|\\
&= 1
\end{aligned}$$

(e)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} 1 & 4\\ 0 & 1\end{bmatrix}\\
|\det(\mathrm{T})| &= |1\times 1-4\times 0|\\
&= 1
\end{aligned}$$

(f)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} 1 & 0\\ 3 & 1\end{bmatrix}\\
|\det(\mathrm{T})| &= |1\times 1-0\times 3|\\
&= 1
\end{aligned}$$

8.
$$\begin{aligned}
\mathrm{A}^2 &= \begin{bmatrix} x & 1\\ 0 & 3\end{bmatrix}\begin{bmatrix} x & 1\\ 0 & 3\end{bmatrix}\\
&= \begin{bmatrix} x^2 & x+3\\ 0 & 9\end{bmatrix}\\
\mathrm{A}^2+\mathrm{A} &= \begin{bmatrix} x^2 & x+3\\ 0 & 9\end{bmatrix}+\begin{bmatrix} x & 1\\ 0 & 3\end{bmatrix}\\
&= \begin{bmatrix} x^2+x & x+4\\ 0 & 12\end{bmatrix}\\
\mathrm{A}^2+\mathrm{A} &= \begin{bmatrix} 6 & x^2-8\\ p & q\end{bmatrix}\\
\therefore\quad \begin{bmatrix} x^2+x & x+4\\ 0 & 12\end{bmatrix} &= \begin{bmatrix} 6 & x^2-8\\ p & q\end{bmatrix}
\end{aligned}$$

From element $(2,1)$, $p=0$.

From element $(2,2)$, $q=12$.

From element $(1,1)$,

$$\begin{aligned}
x^2+x &= 6\\
x^2+x-6 &= 0
\end{aligned}$$

and from element $(1,2)$:

$$\begin{aligned}
x+4 &= x^2-8\\
x^2-x-12 &= 0\\
\text{so}\qquad x^2+x-6 &= x^2-x-12\\
2x &= -6\\
x &= -3
\end{aligned}$$

9. (a)
$$\begin{aligned}
\mathrm{AB} &= \begin{bmatrix} -5-9 & 0-6\\ 1+3 & 0+2\end{bmatrix}\\
&= \begin{bmatrix} -14 & -6\\ 4 & 2\end{bmatrix}
\end{aligned}$$

(b)
$$\begin{aligned}
\mathrm{BA} &= \begin{bmatrix} -5+0 & 3+0\\ 15-2 & -9+2\end{bmatrix}\\
&= \begin{bmatrix} -5 & 3\\ 13 & -7\end{bmatrix}
\end{aligned}$$

(c)
$$\begin{aligned}
\mathrm{A}^{-1} &= \frac{1}{5-3}\begin{bmatrix} 1 & 3\\ 1 & 5\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} 1 & 3\\ 1 & 5\end{bmatrix}
\end{aligned}$$

(d)
$$\begin{aligned}
\mathrm{B}^{-1} &= \frac{1}{-2-0}\begin{bmatrix} 2 & 0\\ -3 & -1\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} -2 & 0\\ 3 & 1\end{bmatrix}
\end{aligned}$$

(e)
$$\begin{aligned}
\mathrm{AC} &= \mathrm{B}\\
\mathrm{C} &= \mathrm{A}^{-1}\mathrm{B}\\
&= \frac{1}{2}\begin{bmatrix} 1 & 3\\ 1 & 5\end{bmatrix}\begin{bmatrix} -1 & 0\\ 3 & 2\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} -1+9 & 0+6\\ -1+15 & 0+10\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} 8 & 6\\ 14 & 10\end{bmatrix}\\
&= \begin{bmatrix} 4 & 3\\ 7 & 5\end{bmatrix}
\end{aligned}$$

(f) $\mathrm{DA} = \mathrm{B}$

$$\mathrm{D} = \mathrm{BA}^{-1}$$
$$= \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} -1+0 & -3+0 \\ 3+2 & 9+10 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} -1 & -3 \\ 5 & 19 \end{bmatrix}$$

Note: the solution in Sadler is incorrectly missing the $\frac{1}{2}$.

10.
$$\mathrm{AA}^{-1} = \mathrm{I}$$
$$\begin{bmatrix} a & -1 & a \\ b & 3 & c \\ d & 1 & -1 \end{bmatrix} \begin{bmatrix} -3 & e & f \\ 1 & 0 & g \\ 4 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} -3a-1+4a & ae+0-a & af-g+7a \\ -3b+3+4c & be+0-c & bf+3g+7c \\ -3d+1-4 & de+0+1 & df+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From matrix element (3,1),

$$-3d-3 = 0$$
$$d = -1$$
$$\begin{bmatrix} a-1 & ae-a & af-g+7a \\ -3b+3+4c & be-c & bf+3g+7c \\ 0 & -e+1 & -f+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from element (3,2),

$$-e+1 = 0$$
$$e = 1$$
$$\begin{bmatrix} a-1 & 0 & af-g+7a \\ -3b+3+4c & b-c & bf+3g+7c \\ 0 & 0 & -f+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from element (1,1),

$$a-1 = 1$$
$$a = 2$$
$$\begin{bmatrix} 1 & 0 & 2f-g+14 \\ -3b+3+4c & b-c & bf+3g+7c \\ 0 & 0 & -f+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from element (2,2),

$$b-c = 1$$
$$c = b-1$$
$$\begin{bmatrix} 1 & 0 & 2f-g+14 \\ -3b+3+4(b-1) & 1 & bf+3g+7(b-1) \\ 0 & 0 & -f+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from element (2,1),

$$-3b+3+4(b-1) = 0$$
$$b-1 = 0$$
$$b = 1$$
$$\text{hence } c = b-1$$
$$c = 0$$
$$\begin{bmatrix} 1 & 0 & 2f-g+14 \\ 0 & 1 & f+3g \\ 0 & 0 & -f+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from elements (2,3) and (3,2),

$$f+3g = 0$$
$$-f+g-7 = 1$$
$$-f+g = 8$$
$$4g = 8$$
$$g = 2$$
$$-f+2 = 8$$
$$f = -6$$

Check that this works for the remaining element: $2(-6)-2+14 = 0$ is correct.

Therefore, $a = 2$, $b = 1$, $c = 0$, $d = -1$, $e = 1$, $f = -6$ and $g = 2$.

11. Let the transition matrix T be defined as

$$\begin{array}{cc} & \text{Received by} \\ & \begin{array}{ccc} \mathrm{P} & \mathrm{M} & \mathrm{T} \end{array} \\ \text{Thrown by} \begin{array}{c} \text{Phillipe} \\ \text{Marlon} \\ \text{Tony} \end{array} & \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \\ 0.7 & 0.3 & 0 \end{bmatrix} \end{array}$$

There are two ways we can find the long term percentage of passes each will receive. (Note that the percentage of passes each receives is equal to the percentage throws each gives and equal to the percentage each has possession of the ball.)

First, empirically. Suppose (without loss of generality) that Phillipe starts with the ball. This gives us an initial state matrix

$$\mathrm{S}_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Now after n passes, the probability of each having the ball is given by

$$\mathrm{S}_n = \mathrm{S}_0\mathrm{T}^n$$

Calculate this for increasingly large values of n until there is no significant change and interpret the results.

After 20 throws,

$$\mathrm{S}_0\mathrm{T}^{20} = \begin{bmatrix} 0.374 & 0.317 & 0.308 \end{bmatrix}$$

Phillipe received 37% of passes, Marlon 32% and Tony 31%.

The more analytical approach is to find the state matrix S such that $\mathrm{ST} = \mathrm{S}$, that is

$$\begin{bmatrix} p & m & t \end{bmatrix} \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \\ 0.7 & 0.3 & 0 \end{bmatrix} = \begin{bmatrix} p & m & t \end{bmatrix}$$
$$\begin{bmatrix} 0.5m+0.7t & 0.6p+0.3t & 0.4p+0.5m \end{bmatrix} = \begin{bmatrix} p & m & t \end{bmatrix}$$
$$-p+0.5m+0.7t = 0$$
$$0.6p-m+0.3t = 0$$
$$0.4p+0.5m-t = 0$$

Solve any two of these simultaneously together with

$$p+m+t = 1$$

to give the same answers as we found empirically. This second approach can be used to find exact values, but this is seldom of relevance with processes that are probabilistic in nature.

12. $$\frac{27}{i} = -27i$$
$$= 27\,\text{cis}(-\frac{\pi}{2})$$

This gives us a principal solution of

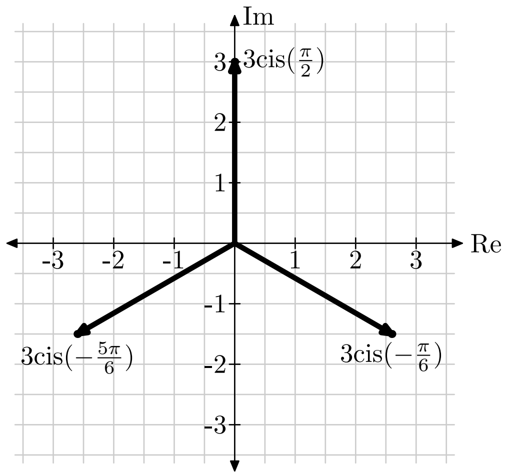
$$z = 3\,\text{cis}(-\frac{\pi}{6})$$

The three solutions are separated by $\frac{2\pi}{3}$ so the other two solutions are

$$z = 3\,\text{cis}(-\frac{\pi}{6} - \frac{2\pi}{3}) = 3\,\text{cis}(-\frac{5\pi}{6})$$

and

$$z = 3\,\text{cis}(-\frac{\pi}{6} + \frac{2\pi}{3}) = 3\,\text{cis}(\frac{\pi}{2})$$



13. (a) The Leslie matrix is

$$L = \begin{bmatrix} 0 & 1.2 & 0.9 \\ 0.6 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix}$$

If P is the population vector from the previous year and Q is the current population, then

$$Q = \begin{bmatrix} 2778 \\ 1572 \\ 1192 \end{bmatrix}$$
$$LP = Q$$
$$L^{-1}LP = L^{-1}Q$$
$$P = \begin{bmatrix} 2620 \\ 1490 \\ 1100 \end{bmatrix}$$

(using calculator for the last step).

Alternatively, we can simply work backward. Second generation population is given by the survival from the previous year's first generation: $q_2 = 0.6p_1$ so $p_1 = \frac{q_2}{0.6} = \frac{1572}{0.6} = 2620$.

Similarly third generation population is given by the survival from the previous year's second generation: $q_3 = 0.8p_2$ so $p_2 = \frac{q_3}{0.8} = \frac{1192}{0.8} = 1490$.

The first generation population is given by the reproduction rates from the previous year:

$$\begin{aligned} q_1 &= 1.2p_2 + 0.9p_3 \\ p_3 &= \frac{q_1 - 1.2p_2}{0.9} \\ &= \frac{2778 - 1.2 \times 1490}{0.9} \\ &= 1100 \end{aligned}$$

(b) The Leslie matrix becomes

$$L = \begin{bmatrix} 0 & 1.2 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix}$$

This is not an invertable matrix so it is not possible to determine the previous year's population.

It *is* possible to work out the first and second generation population by working backward, as previously, but we can tell nothing about the third generation population.

$q_2 = 0.6p_1$ so $p_1 = \frac{q_2}{0.6} = \frac{1572}{0.6} = 2620$.

$q_3 = 0.8p_2$ so $p_2 = \frac{q_3}{0.8} = \frac{1852}{0.8} = 2315$.

The reproduction rate, however, now has no contribution from p_3: $q_1 = 1.2p_2$ which gives $p_2 = 2315$ which is consistent with the value given by the survival rates, but tells us nothing about p_3.

14. (a) $$\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

(b) $$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

(c) The gradient of the line of symmetry gives

$$\begin{aligned} \tan\theta &= \frac{\sqrt{3}}{3} \\ \theta &= 30° \\ 2\theta &= 60° \end{aligned}$$

so the transformation matrix is

$$\begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

(d) The transformation matrix representing the combined reflection and rotation described is

$$
\begin{aligned}
&\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} \sqrt{3}+\sqrt{3} & -1+3 \\ 3-1 & -\sqrt{3}-\sqrt{3} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}
\end{aligned}
$$

and this is not equal to the transformation matrix representing a 60° anticlockwise rotation about the origin.

The apparent equivalence in the diagram is caused by the symmetry of the square. (Refer to the solutions in Sadler for a more full explanation.)

15. $$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}2ex\ln x &= 2e\ln x + 2ex\frac{1}{x} \\
&= 2e\ln x + 2e \\
&= 2e(\ln(x)+1)
\end{aligned}
$$

At the stationary point

$$
\begin{aligned}
2e(\ln(x)+1) &= 0 \\
\ln x &= -1 \\
x &= e^{-1} \\
&= \frac{1}{e}
\end{aligned}
$$

The second derivative is

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2}2ex\ln x &= \frac{\mathrm{d}}{\mathrm{d}x}2e(\ln(x)+1) \\
&= \frac{2e}{x}
\end{aligned}
$$

and at $x = \frac{1}{e}$

$$
\frac{\mathrm{d}^2}{\mathrm{d}x^2}2ex\ln x = 2e^2
$$

which is positive, signifying that the gradient is increasing, so the stationary point is a minimum.

The minimum value is obtained by substituting $x = \frac{1}{e}$ into the original expression:

$$
\begin{aligned}
2e(\frac{1}{e})\ln\frac{1}{e} &= 2\ln e^{-1} \\
&= -2
\end{aligned}
$$

16. $$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{2}{x}(x^2) - 2\ln x(2x)}{x^4} \\
&= \frac{2x - 4x\ln x}{x^4} \\
&= \frac{2x(1-2\ln x)}{x^4} \\
&= \frac{2(1-2\ln x)}{x^3}
\end{aligned}
$$

At the stationary points

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{2(1-2\ln x)}{x^3} &= 0 \\
1 - 2\ln x &= 0 \\
\ln x &= \frac{1}{2} \\
x &= e^{\frac{1}{2}} \\
&= \sqrt{e} \\
y &= \frac{2\ln e^{\frac{1}{2}}}{e} \\
&= \frac{1}{e}
\end{aligned}
$$

There is one stationary point, a maximum at $(\sqrt{e}, \frac{1}{e})$.

17. (a) De Moivre's theorem states that

$$
(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta
$$

so given

$$
\begin{aligned}
z &= \cos\theta + \mathrm{i}\sin\theta \\
z^k &= \cos k\theta + \mathrm{i}\sin k\theta
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{z^k} &= z^{-k} \\
&= \cos(-k\theta) + \mathrm{i}\sin(-k\theta) \\
&= \cos k\theta - \mathrm{i}\sin k\theta \\
\therefore \quad z^k + \frac{1}{z^k} &= \cos k\theta + \mathrm{i}\sin k\theta \\
&\quad + \cos k\theta - \mathrm{i}\sin k\theta \\
&= 2\cos k\theta
\end{aligned}
$$

□

(b) i. To prove:

$$
\cos^3\theta = \frac{\cos(3\theta) + 3\cos\theta}{4}
$$

Proof:

$$
\begin{aligned}
\text{LHS} &= \cos^3\theta \\
&= \left(\frac{z+\frac{1}{z}}{2}\right)^3 \\
&= \frac{z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}}{8} \\
&= \frac{z^3 + \frac{1}{z^3} + 3z + \frac{3}{z}}{8} \\
&= \frac{\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)}{8} \\
&= \frac{2\cos(3\theta) + 6\cos\theta}{8} \\
&= \frac{\cos(3\theta) + 3\cos\theta}{4} \\
&= \text{RHS}
\end{aligned}
$$

□

ii. To prove:

$$\cos^4\theta = \frac{\cos(4\theta) + 4\cos(2\theta) + 3}{8}$$

Proof:

$$\begin{aligned}
\text{LHS} &= \cos^4\theta \\
&= \left(\frac{z + \frac{1}{z}}{2}\right)^4 \\
&= \frac{z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}}{16} \\
&= \frac{z^4 + \frac{1}{z^4} + 4z^2 + \frac{4}{z^2} + 6}{16} \\
&= \frac{\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6}{16} \\
&= \frac{2\cos(4\theta) + 8\cos(2\theta) + 6}{16} \\
&= \frac{\cos(4\theta) + 4\cos(2\theta) + 3}{8} \\
&= \text{RHS}
\end{aligned}$$

□