Chapter 3

Exercise 3A

1. The calculator display given shows the 1st quadrant solution. There will also be a 2nd quadrant solution at $x = 180 - 14.47751219$ giving two solutions: $x \approx 14.5°$ and $x \approx 165.5°$

2. $\sin x = \pm\frac{1}{2}$ which has solutions in all four quadrants: $x = \frac{\pi}{6}$, $x = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$, $x = -\frac{\pi}{6}$, $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

3. $$2x = \frac{\pi}{6} \qquad \text{or} \qquad 2x = \pi - \frac{\pi}{6}$$
$$x = \frac{\pi}{12} \qquad\qquad = \frac{5\pi}{6}$$
$$x = \frac{5\pi}{12}$$
$$\text{or} \quad 2x = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6}, \quad x = \frac{13\pi}{12} \qquad \text{or} \quad 2x = 3\pi - \frac{\pi}{6} = \frac{17\pi}{6}, \quad x = \frac{17\pi}{12}$$

4. $\sin^2 x + \cos^2 x = 1$ so this question simplifies to $\sin x = 1$ with solution $x = \frac{\pi}{2}$.

5. Use the null factor law:
$$\begin{aligned} 2\sin x - 1 &= 0 \\ 2\sin x &= 1 \\ \sin x &= \frac{1}{2} \\ x &= \frac{\pi}{6} \\ \text{or } x &= \pi - \frac{\pi}{6} \\ &= \frac{5\pi}{6} \end{aligned} \qquad \text{or} \qquad \begin{aligned} \cos x &= 0 \\ x &= \frac{\pi}{2} \\ \text{or } x &= \frac{3\pi}{2} \end{aligned}$$

6. First factorise:
$$\begin{aligned} \sin x + 2\sin^2 x &= 0 \\ \sin x(1 + 2\sin x) &= 0 \end{aligned}$$
Now use the null factor law:
$$\begin{aligned} \sin x &= 0 \\ x &= 0 \\ \text{or } x &= 180° \\ \text{or } x &= 360° \end{aligned} \qquad \text{or} \qquad \begin{aligned} 1 + 2\sin x &= 0 \\ 2\sin x &= -1 \\ \sin x &= -\frac{1}{2} \\ x &= 180 + 30 \\ &= 210° \\ \text{or } x &= 360 - 30 \\ &= 330° \end{aligned}$$

7. Use the null factor law:
$$\begin{aligned} 2\cos x + 1 &= 0 \\ 2\cos x &= -1 \\ \cos x &= -\frac{1}{2} \end{aligned} \qquad \begin{aligned} 5\sin x - 1 &= 0 \\ 5\sin x &= 1 \\ \sin x &= \frac{1}{5} \end{aligned}$$
The first factor will have solutions in the second and third quadrant; the second will have solutions in the first and second quadrants.
$$\begin{aligned} x &= 180 - 60 \\ &= 120° \\ \text{or } x &= 180 + 60 \\ &= 240° \end{aligned} \qquad \begin{aligned} x &= 11.5° \\ \text{or } x &= 180 - 11.5 \\ &= 168.5° \end{aligned}$$

8. First use the pythagorean identity to replace $\cos^2 x$ with $\sin^2 x$:
$$\begin{aligned} \sin x + \left(\sqrt{2}\right)\cos^2 x &= \sqrt{2} \\ \sin x + \left(\sqrt{2}\right)(1 - \sin^2 x) &= \sqrt{2} \\ \sin x + \sqrt{2} - \left(\sqrt{2}\right)\sin^2 x &= \sqrt{2} \\ \sin x - \left(\sqrt{2}\right)\sin^2 x &= 0 \\ \sin x\left(1 - \left(\sqrt{2}\right)\sin x\right) &= 0 \end{aligned}$$
Now use the null factor law:
$$\begin{aligned} \sin x &= 0 \\ x &= 0 \\ \text{or } x &= -\pi \\ \text{or } x &= \pi \end{aligned} \qquad \text{or} \qquad \begin{aligned} 1 - \left(\sqrt{2}\right)\sin x &= 0 \\ \left(\sqrt{2}\right)\sin x &= 1 \\ \sin x &= \frac{1}{\sqrt{2}} \\ x &= \frac{\pi}{4} \\ \text{or } x &= \pi - \frac{\pi}{4} \\ &= \frac{3\pi}{4} \end{aligned}$$

9.
$$\begin{aligned}8\sin^2 x + 4\cos^2 x &= 7\\4\sin^2 x + (4\sin^2 x + 4\cos^2 x) &= 7\\4\sin^2 x + 4(\sin^2 x + \cos^2 x) &= 7\\4\sin^2 x + 4 &= 7\\4\sin^2 x &= 3\\\sin^2 x &= \frac{3}{4}\\\sin x &= \pm\frac{\sqrt{3}}{2}\\x &= \frac{\pi}{3}\\\text{or } x &= \pi - \frac{\pi}{3}\\&= \frac{2\pi}{3}\\\text{or } x &= \pi + \frac{\pi}{3}\\&= \frac{4\pi}{3}\\\text{or } x &= 2\pi - \frac{\pi}{3}\\&= \frac{5\pi}{3}\end{aligned}$$

10. Rearrange and factorise:

$$\begin{aligned}\tan^2 x + \tan x &= 2\\\tan^2 x + \tan x - 2 &= 0\\(\tan x + 2)(\tan x - 1) &= 0\end{aligned}$$

Null factor law:

$$\begin{aligned}\tan x + 2 &= 0 & \tan x - 1 &= 0\\\tan x &= -2 & \tan x &= 1\end{aligned}$$

The first factor has solutions in the 2nd and 4th quadrants. The second factor has solutions in the 1st and 3rd quadrants.

$$\begin{aligned}x &= 180 - 63.4 & x &= 45^\circ\\&= 116.6^\circ & &\\\text{or } x &= -63.4 & x &= -180 + 45\\& & &= -135^\circ\end{aligned}$$

11.
$$\begin{aligned}\sqrt{3}\sin x - 2\cos^2 x + 2 &= 0\\\sqrt{3}\sin x - 2(1 - \sin^2 x) + 2 &= 0\\\sqrt{3}\sin x - 2 + 2\sin^2 x + 2 &= 0\\\sqrt{3}\sin x + 2\sin^2 x &= 0\end{aligned}$$

Null factor law:

$$\begin{aligned}\sin x &= 0 & \sqrt{3} + 2\sin x &= 0\\x &= 0 & 2\sin x &= -\sqrt{3}\\\text{or } x &= \pi & \sin x &= -\frac{\sqrt{3}}{2}\\\text{or } x &= 2\pi & x &= \pi + \frac{\pi}{3}\\& & &= \frac{4\pi}{3}\\& & \text{or } x &= 2\pi - \frac{\pi}{3}\\& & &= \frac{5\pi}{3}\end{aligned}$$

12.
$$\begin{aligned}5 - 4\cos x &= 4\sin^2 x\\5 - 4\cos x &= 4(1 - \cos^2 x)\\5 - 4\cos x &= 4 - 4\cos^2 x\\1 - 4\cos x &= -4\cos^2 x\\4\cos^2 x - 4\cos x + 1 &= 0\\(2\cos x - 1)^2 &= 0\\2\cos x - 1 &= 0\\2\cos x &= 1\\\cos x &= \frac{1}{2}\\x &= \pm 60^\circ\end{aligned}$$

13.
$$\begin{aligned}3 &= 2\cos^2 x + 3\sin x\\3 &= 2(1 - \sin^2 x) + 3\sin x\\3 &= 2 - 2\sin^2 x + 3\sin x\\2\sin^2 x - 3\sin x + 1 &= 0\\(\sin x - 1)(2\sin x - 1) &= 0\end{aligned}$$

$$\begin{aligned}\sin x - 1 &= 0 & 2\sin x - 1 &= 0\\\sin x &= 1 & 2\sin x &= 1\\x &= \frac{\pi}{2} & \sin x &= \frac{1}{2}\\\text{or } x &= 2\pi + \frac{\pi}{2} & x &= \frac{\pi}{6}\\&= \frac{5\pi}{2} & \text{or } x &= \pi - \frac{\pi}{6}\\& & &= \frac{5\pi}{6}\\& & \text{or } x &= 2\pi + \frac{\pi}{6}\\& & &= \frac{13\pi}{6}\\& & \text{or } x &= 3\pi - \frac{\pi}{6}\\& & &= \frac{17\pi}{6}\end{aligned}$$

14.
$$\begin{aligned}(\sin x)(2+\sin x)+\cos^2 x &= 0\\ 2\sin x+\sin^2 x+\cos^2 x &= 0\\ 2\sin x+1 &= 0\\ 2\sin x &= -1\\ \sin x &= -\frac{1}{2}\\ x &= \pi+\frac{\pi}{6}\\ &= \frac{7\pi}{6}\\ \text{or } x &= 2\pi-\frac{\pi}{6}\\ &= \frac{11\pi}{6}\end{aligned}$$

15.
$$\begin{aligned}2\cos x-\sqrt{3}\cos x\sin x &= 0\\ \cos x(2-\sqrt{3}\sin x &= 0\end{aligned}$$

$$\begin{aligned}\cos x &= 0 & 2-\sqrt{3}\sin x &= 0\\ x &= -\frac{\pi}{2} & \sqrt{3}\sin x &= 2\\ \text{or } x &= \frac{\pi}{2} & \sin x &= \frac{2}{\sqrt{3}}\end{aligned}$$

(no real solution)

16.
$$\begin{aligned}(\sin x)(1-\sin x) &= -\cos^2 x\\ (\sin x)(1-\sin x) &= -(1-\sin^2 x)\\ (\sin x)(1-\sin x) &= -1+\sin^2 x\\ \sin x-\sin^2 x &= -1+\sin^2 x\\ 2\sin^2 x-\sin x-1 &= 0\\ (2\sin x+1)(\sin x-1) &= 0\end{aligned}$$

$$\begin{aligned}2\sin x+1 &= 0 & \sin x-1 &= 0\\ 2\sin x &= -1 & \sin x &= 1\\ \sin x &= -\frac{1}{2} & x &= \frac{\pi}{2}\\ x &= \pi+\frac{\pi}{6}\\ &= \frac{7\pi}{6}\end{aligned}$$

17.
$$\begin{aligned}\sin x\tan x &= 2-\cos x\\ \sin x\frac{\sin x}{\cos x} &= 2-\cos x\\ \frac{\sin^2 x}{\cos x} &= 2-\cos x\\ \sin^2 x &= \cos x(2-\cos x)\\ \sin^2 x &= 2\cos x-\cos^2 x\\ 1-\cos^2 x &= 2\cos x-\cos^2 x\\ 1 &= 2\cos x\\ \cos x &= \frac{1}{2}\\ x &= \frac{\pi}{3}\\ \text{or } x &= -\frac{\pi}{3}\\ \text{or } x &= 2\pi-\frac{\pi}{3}\\ &= \frac{5\pi}{3}\text{or } x \quad = -2\pi+\frac{\pi}{3}\\ &= \frac{-5\pi}{3}\end{aligned}$$

18. L.H.S.:

$$\begin{aligned}2\cos^2\theta+3 &= 2(1-\sin^2\theta)+3\\ &= 2-2\sin^2\theta+3\\ &= 5-2\sin^2\theta\\ &= \text{R.H.S.}\end{aligned}$$

□

19. L.H.S.:

$$\begin{aligned}\sin\theta-\cos^2\theta &= \sin\theta-(1-\sin^2\theta)\\ &= \sin\theta-1+\sin^2\theta\\ &= \sin\theta+\sin^2\theta-1\\ &= (\sin\theta)(1+\sin\theta)-1\\ &= \text{R.H.S.}\end{aligned}$$

□

20. L.H.S.:

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= 2\sin\theta\cos\theta+\sin^2\theta+\cos^2\theta\\ &= 2\sin\theta\cos\theta+1\\ &= \text{R.H.S.}\end{aligned}$$

□

21. R.H.S.:

$$\begin{aligned}(\sin\theta-\cos\theta)^2 &= \sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta\\ &= \sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta\\ &= 1-2\sin\theta\cos\theta\\ &= \text{L.H.S.}\end{aligned}$$

□

22. L.H.S.:

$$\sin^4\theta - \cos^4\theta = (\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta)$$

(difference of perfect squares)

$$\begin{aligned} &= 1(\sin^2\theta - \cos^2\theta) \\ &= (1-\cos^2\theta) - \cos^2\theta \\ &= 1 - 2\cos^2\theta \\ &= \text{R.H.S.} \end{aligned}$$

□

23. L.H.S.:

$$\begin{aligned} \sin^4\theta - \sin^2\theta &= \sin^2\theta(\sin^2\theta - 1) \\ &= (1-\cos^2\theta)(-\cos^2\theta) \\ &= -\cos^2\theta + \cos^4\theta \\ &= \cos^4\theta - \cos^2\theta \\ &= \text{R.H.S.} \end{aligned}$$

□

24. L.H.S.:

$$\begin{aligned} \sin^2\theta\tan^2\theta &= (1-\cos^2\theta)\tan^2\theta \\ &= \tan^2\theta - \cos^2\theta\tan^2\theta \\ &= \tan^2\theta - \cos^2\theta\frac{\sin^2\theta}{\cos^2\theta} \\ &= \tan^2\theta - \sin^2\theta \\ &= \text{R.H.S.} \end{aligned}$$

□

25. L.H.S.:

$$\begin{aligned} (1+\sin\theta)(1-\sin\theta) &= 1 - \sin^2\theta \\ &= \cos^2\theta \\ &= 1 + \cos^2\theta - 1 \\ &= 1 + (\cos\theta + 1)(\cos\theta - 1) \\ &= \text{R.H.S.} \end{aligned}$$

□

26. L.H.S.:

$$\begin{aligned} \sin\theta\tan\theta + \cos\theta &= \sin\theta\frac{\sin\theta}{\cos\theta} + \cos\theta \\ &= \frac{\sin^2\theta}{\cos\theta} + \cos\theta \\ &= \frac{1-\cos^2\theta}{\cos\theta} + \cos\theta \\ &= \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta} + \cos\theta \\ &= \frac{1}{\cos\theta} - \cos\theta + \cos\theta \\ &= \frac{1}{\cos\theta} \\ &= \text{R.H.S.} \end{aligned}$$

□

27. L.H.S.:

$$\begin{aligned} \frac{1}{1+\tan^2\theta} &= \frac{1}{1+\frac{\sin^2\theta}{\cos^2\theta}} \\ &= \frac{1}{\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}} \\ &= \frac{1}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}} \\ &= \frac{\cos^2\theta}{\cos^2\theta + \sin^2\theta} \\ &= \frac{\cos^2\theta}{1} \\ &= \cos^2\theta \\ &= \text{R.H.S.} \end{aligned}$$

□

28. R.H.S.:

$$\begin{aligned} \frac{1+\cos\theta}{1-\cos\theta} &= \frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta} \\ &= \frac{1+2\cos\theta+\cos^2\theta}{1-\cos^2\theta} \\ &= \frac{\cos^2\theta + 2\cos\theta + 1}{\sin^2\theta} \\ &= \text{L.H.S.} \end{aligned}$$

□

29. L.H.S.:

$$\begin{aligned} \frac{\sin\theta}{1-\cos\theta} - \frac{\cos\theta}{\sin\theta} &= \frac{\sin\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta} - \frac{\cos\theta}{\sin\theta} \\ &= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} - \frac{\cos\theta}{\sin\theta} \\ &= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} - \frac{\cos\theta}{\sin\theta} \\ &= \frac{1+\cos\theta}{\sin\theta} - \frac{\cos\theta}{\sin\theta} \\ &= \frac{1+\cos\theta-\cos\theta}{\sin\theta} \\ &= \frac{1}{\sin\theta} \\ &= \text{R.H.S.} \end{aligned}$$

□

30. L.H.S.:

$$\begin{aligned} \frac{1-\sin\theta\cos\theta-\cos^2\theta}{\sin^2\theta+\sin\theta\cos\theta-1} &= \frac{1-\sin\theta\cos\theta-(1-\sin^2\theta)}{(1-\cos^2\theta)+\sin\theta\cos\theta-1} \\ &= \frac{1-\sin\theta\cos\theta-1+\sin^2\theta}{1-\cos^2\theta+\sin\theta\cos\theta-1} \\ &= \frac{-\sin\theta\cos\theta+\sin^2\theta}{-\cos^2\theta+\sin\theta\cos\theta} \\ &= \frac{\sin\theta(-\cos\theta+\sin\theta)}{\cos\theta(-\cos\theta+\sin\theta)} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta \\ &= \text{R.H.S.} \end{aligned}$$

□

Exercise 3B

1. Compare this with identity ④ on page 57 of Sadler. Substitute $A = 2x$ and $B = x$:

$$\begin{aligned}\sin 2x \cos x + \cos 2x \sin x &= \sin(2x + x)\\ &= \sin 3x\end{aligned}$$

2. Compare this with identity ① on page 56 of Sadler. Substitute $A = 3x$ and $B = x$:

$$\begin{aligned}\cos 3x \cos x + \sin 3x \sin x &= \cos(3x - x)\\ &= \cos 2x\end{aligned}$$

3. Compare this with identity ③ on page 57 of Sadler. Substitute $A = 5x$ and $B = x$:

$$\begin{aligned}\sin 5x \cos x - \cos 5x \sin x &= \sin(5x - x)\\ &= \sin 4x\end{aligned}$$

4. Compare this with identity ② on page 56 of Sadler. Substitute $A = 7x$ and $B = x$:

$$\begin{aligned}\cos 7x \cos x - \sin 7x \sin x &= \cos(7x + x)\\ &= \cos 8x\end{aligned}$$

5. $$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

6. $$\begin{aligned}\tan 15^\circ &= \tan(45^\circ - 30^\circ)\\ &= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}\\ &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \times \frac{1}{\sqrt{3}}}\end{aligned}$$

7. $$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

   Compare this with the answer for question 5. We could have arrived at this more simply using the identity $\sin(90^\circ - A) = \cos(A)$ substituting $A = 15^\circ$.

8. $$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

9. $$\begin{aligned}\tan 75^\circ &= \tan(45^\circ + 30^\circ)\\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}\\ &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \times \frac{1}{\sqrt{3}}}\\ &= \frac{\frac{\sqrt{3}}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{\frac{\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}}}\\ &= \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\\ &= \frac{3 + 2\sqrt{3} + 1}{3 - 1}\\ &= \frac{4 + 2\sqrt{3}}{2}\\ &= 2 + \sqrt{3}\end{aligned}$$

10. $$\begin{aligned}2\sin(\theta + 45^\circ) &= 2(\sin\theta \cos 45^\circ + \cos\theta \sin 45^\circ)\\ &= 2\left(\frac{1}{\sqrt{2}} \sin\theta + \frac{1}{\sqrt{2}} \cos\theta\right)\\ &= \frac{2}{\sqrt{2}} \sin\theta + \frac{2}{\sqrt{2}} \cos\theta\\ &= \sqrt{2}\sin\theta + \sqrt{2}\cos\theta\end{aligned}$$

    $a = b = \sqrt{2}$

11. $$8\cos\left(\theta - \frac{\pi}{3}\right) = 8\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right)$$
$$= 8\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)$$
$$= 4\cos\theta + 4\sqrt{3}\sin\theta$$
$$= 4\sqrt{3}\sin\theta + 4\cos\theta$$

$c = 4\sqrt{3},\ d = 4$

12. $$4\cos(\theta + 30^\circ) = 4(\cos\theta\cos 30^\circ - \sin\theta\sin 30^\circ)$$
$$= 4\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)$$
$$= 2\sqrt{3}\cos\theta - 2\sin\theta$$

$e = 2\sqrt{3},\ f = -2$

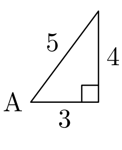
13. $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$= \frac{5\sqrt{3} - \frac{\sqrt{3}}{4}}{1 - 5\sqrt{3} \times -\frac{\sqrt{3}}{4}}$$
$$= \frac{\sqrt{3}(5 - \frac{1}{4})}{1 + 5 \times \frac{3}{4}}$$
$$= \frac{\frac{19\sqrt{3}}{4}}{1 + \frac{15}{4}}$$
$$= \frac{\frac{19\sqrt{3}}{4}}{\frac{19}{4}}$$
$$= \sqrt{3}$$

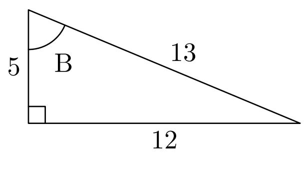
$\tan(A+B)$ is positive, so $A+B$ is in quadrant 3: $A + B = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

14. To proceed, we need to know $\cos A$ and $\sin B$. Given that we are working with acute angles, $\cos A = \sqrt{1 - \sin^2 A}$. (If the angle was not known to be acute we'd have to also consider $\cos A = -\sqrt{1 - \sin^2 A}$.) This gives us

$$\cos A = \sqrt{1 - \left(\frac{4}{5}\right)} \qquad \sin B = \sqrt{1 - \left(\frac{5}{13}\right)}$$
$$= \frac{3}{5} \qquad\qquad = \frac{12}{13}$$

Another way of looking at this is to think about the given values in the context of a right angle triangle. $\sin \mathrm{A} = \frac{4}{5}$ so think of a triangle with hypotenuse of 5 units and opposite of 4 units. Pythagoras' theorem gives us 3 for the other side resulting in $\cos \mathrm{A} = \frac{3}{5}$ and $\tan \mathrm{A} = \frac{4}{3}$. Similarly, given $\cos \mathrm{B} = \frac{5}{13}$ think of a triangle with hypotenuse of 13 and adjacent of 5; Pythagoras gives us 12 as the remaining side and hence $\sin \mathrm{B} = \frac{12}{13}$ and $\tan \mathrm{B} = \frac{12}{5}$.





(a) $$\sin(\mathrm{A} + \mathrm{B}) = \sin \mathrm{A}\cos \mathrm{B} + \cos \mathrm{A}\sin \mathrm{B}$$
$$= \frac{4}{5} \times \frac{5}{13} + \frac{3}{5} \times \frac{12}{13}$$
$$= \frac{20}{65} + \frac{36}{65}$$
$$= \frac{56}{65}$$

(b) $$\cos(\mathrm{A} - \mathrm{B}) = \cos \mathrm{A}\cos \mathrm{B} + \sin \mathrm{A}\sin \mathrm{B}$$
$$= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13}$$
$$= \frac{15}{65} + \frac{48}{65}$$
$$= \frac{63}{65}$$

15. Pythagoras (see question 14 above) gives us $\cos \mathrm{D} = \frac{24}{25}$ and $\cos \mathrm{E} = \frac{4}{5}$.

(a) $$\sin(\mathrm{D} - \mathrm{E}) = \sin \mathrm{D}\cos \mathrm{E} - \cos \mathrm{D}\sin \mathrm{E}$$
$$= \frac{7}{25} \times \frac{4}{5} - \frac{24}{25} \times \frac{3}{5}$$
$$= \frac{28}{125} - \frac{72}{125}$$
$$= -\frac{44}{125}$$

(b) $$\cos(\mathrm{D} + \mathrm{E}) = \cos \mathrm{D}\cos \mathrm{E} - \sin \mathrm{D}\sin \mathrm{E}$$
$$= \frac{24}{25} \times \frac{4}{5} - \frac{7}{25} \times \frac{3}{5}$$
$$= \frac{96}{125} - \frac{21}{125}$$
$$= \frac{75}{125}$$
$$= \frac{3}{5}$$

16. To prove: $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

Proof:

L.H.S.: $$\sin\left(x + \frac{\pi}{2}\right) = \sin x\cos\frac{\pi}{2} + \cos x\sin\frac{\pi}{2}$$
$$= \sin x \times 0 + \cos x \times 1$$
$$= \cos x$$
$$= \text{R.H.S}$$

□

17. (a) To prove: $\sin(x + 2\pi) = \sin x$

Proof:

$$\text{L.H.S.} = \sin(x + 2\pi)$$
$$= \sin x\cos 2\pi + \cos x\sin 2\pi$$
$$= \sin x \times 1 + \cos x \times 0$$
$$= \sin x$$
$$= \text{R.H.S}$$

□

(b) To prove: $\sin(x - 2\pi) = \sin x$
Proof:

$$\begin{aligned}\text{L.H.S.} &= \sin(x - 2\pi)\\ &= \sin x \cos 2\pi - \cos x \sin 2\pi\\ &= \sin x \times 1 - \cos x \times 0\\ &= \sin x\\ &= \text{R.H.S}\end{aligned}$$

□

18. To prove: $\cos(x + 2\pi) = \cos x$

Proof:

$$\begin{aligned}\text{L.H.S.} &= \cos(x + 2\pi)\\ &= \cos x \cos 2\pi - \sin x \sin 2\pi\\ &= \cos x \times 1 + \sin x \times 0\\ &= \cos x\\ &= \text{R.H.S}\end{aligned}$$

□

19. To prove: $\tan(x + \pi) = \tan x$

Proof:

$$\begin{aligned}\text{L.H.S.:}\quad \tan(x + \pi) &= \frac{\tan x + \tan \pi}{1 - \tan x \tan \pi}\\ &= \frac{\tan x + 0}{1 - \tan x \times 0}\\ &= \frac{\tan x}{1}\\ &= \tan x\\ &= \text{R.H.S}\end{aligned}$$

□

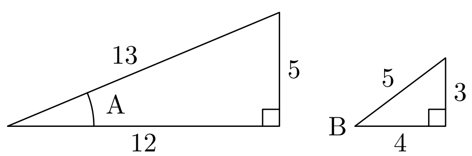
20. To prove: $\tan(-x) = -\tan x$

Proof:

$$\begin{aligned}\text{L.H.S.:}\quad \tan(-x) &= \tan(0 - x)\\ &= \frac{\tan 0 - \tan x}{1 + \tan 0 \tan x}\\ &= \frac{-\tan x}{1 + 0 \times \tan x}\\ &= \frac{-\tan x}{1}\\ &= -\tan x\\ &= \text{R.H.S}\end{aligned}$$

□

21. Given that they are obtuse angles, A and B fall into the 2nd quadrant so their sines are positive and cosines and tangents negative. Use Pythagoras to find the necessary ratios from those given using the triangle approach outlined in question 14 above:



θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
A	$\frac{5}{13}$	$-\frac{12}{13}$	$-\frac{5}{12}$
B	$\frac{3}{5}$	$-\frac{4}{5}$	$-\frac{3}{4}$

(a)
$$\begin{aligned}\sin(\text{A} + \text{B}) &= \sin \text{A} \cos \text{B} + \cos \text{A} \sin \text{B}\\ &= \frac{5}{13} \times \left(-\frac{4}{5}\right) + \left(-\frac{12}{13}\right) \times \frac{3}{5}\\ &= -\frac{20}{65} - \frac{36}{65}\\ &= -\frac{56}{65}\end{aligned}$$

(b)
$$\begin{aligned}\cos(\text{A} - \text{B}) &= \cos \text{A} \cos \text{B} + \sin \text{A} \sin \text{B}\\ &= \left(-\frac{12}{13}\right) \times \left(-\frac{4}{5}\right) + \frac{5}{13} \times \frac{3}{5}\\ &= \frac{48}{65} + \frac{15}{65}\\ &= \frac{63}{65}\end{aligned}$$

(c)
$$\begin{aligned}\tan(\text{A} + \text{B}) &= \frac{\tan \text{A} + \tan \text{B}}{1 - \tan \text{A} \tan \text{B}}\\ &= \frac{-\frac{5}{12} - \frac{3}{4}}{1 - \left(-\frac{5}{12}\right)\left(-\frac{3}{4}\right)}\\ &= \frac{\frac{-5-9}{12}}{1 - \frac{5}{16}}\\ &= \frac{\frac{-14}{12}}{\frac{11}{16}}\\ &= -\frac{7}{6} \times \frac{16}{11}\\ &= -\frac{56}{33}\end{aligned}$$

22. To prove: $\sin(\text{A}+\text{B}) - \sin(\text{A}-\text{B}) = 2\cos \text{A} \sin \text{B}$

Proof:

$$\begin{aligned}\text{L.H.S.} &= \sin(\text{A} + \text{B}) - \sin(\text{A} - \text{B})\\ &= \sin \text{A} \cos \text{B} + \cos \text{A} \sin \text{B}\\ &\quad -(\sin \text{A} \cos \text{B} - \cos \text{A} \sin \text{B})\\ &= \sin \text{A} \cos \text{B} + \cos \text{A} \sin \text{B}\\ &\quad - \sin \text{A} \cos \text{B} + \cos \text{A} \sin \text{B}\\ &= 2\cos \text{A} \sin \text{B}\\ &= \text{R.H.S}\end{aligned}$$

□

23. To prove: $\cos(\text{A}-\text{B}) + \cos(\text{A}+\text{B}) = 2\cos \text{A} \cos \text{B}$

Proof:

$$\begin{aligned}\text{L.H.S.} &= \cos(\text{A} - \text{B}) + \sin(\text{A} + \text{B})\\ &= \cos \text{A} \cos \text{B} + \sin \text{A} \sin \text{B}\\ &\quad + \cos \text{A} \cos \text{B} - \sin \text{A} \sin \text{B}\\ &= 2\cos \text{A} \cos \text{B}\\ &= \text{R.H.S}\end{aligned}$$

□

24. To prove: $2\cos\left(x-\frac{\pi}{6}\right)=\sin x+\sqrt{3}\cos x$

Proof:

$$\begin{aligned}\text{L.H.S.} &= 2\cos\left(x-\frac{\pi}{6}\right)\\ &= 2\left(\cos x\cos\frac{\pi}{6}+\sin x\sin\frac{\pi}{6}\right)\\ &= 2\left(\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x\right)\\ &= \sqrt{3}\cos x+\sin x\\ &= \sin x+\sqrt{3}\cos x\\ &= \text{R.H.S}\end{aligned}$$

□

25. To prove: $\tan\left(\theta+\frac{\pi}{4}\right)=\frac{1+\tan\theta}{1-\tan\theta}$

Proof:

$$\begin{aligned}\text{L.H.S.} &= \tan\left(\theta+\frac{\pi}{4}\right)\\ &= \frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\tan\frac{\pi}{4}}\\ &= \frac{\tan\theta+1}{1-\tan\theta\times 1}\\ &= \frac{1+\tan\theta}{1-\tan\theta}\\ &= \text{R.H.S}\end{aligned}$$

□

26. To prove: $\frac{\cos(\text{A}+\text{B})}{\cos(\text{A}-\text{B})}=\frac{1-\tan\text{A}\tan\text{B}}{1+\tan\text{A}\tan\text{B}}$

Proof:

$$\begin{aligned}\text{L.H.S.} &= \frac{\cos(\text{A}+\text{B})}{\cos(\text{A}-\text{B})}\\ &= \frac{\cos\text{A}\cos\text{B}-\sin\text{A}\sin\text{B}}{\cos\text{A}\cos\text{B}+\sin\text{A}\sin\text{B}}\\ &= \frac{\frac{\cos\text{A}\cos\text{B}-\sin\text{A}\sin\text{B}}{\cos\text{A}\cos\text{B}}}{\frac{\cos\text{A}\cos\text{B}+\sin\text{A}\sin\text{B}}{\cos\text{A}\cos\text{B}}}\\ &= \frac{1-\frac{\sin\text{A}\sin\text{B}}{\cos\text{A}\cos\text{B}}}{1+\frac{\sin\text{A}\sin\text{B}}{\cos\text{A}\cos\text{B}}}\\ &= \frac{1-\frac{\sin\text{A}}{\cos\text{A}}\times\frac{\sin\text{B}}{\cos\text{B}}}{1+\frac{\sin\text{A}}{\cos\text{A}}\times\frac{\sin\text{B}}{\cos\text{B}}}\\ &= \frac{1-\tan\text{A}\tan\text{B}}{1+\tan\text{A}\tan\text{B}}\\ &= \text{R.H.S}\end{aligned}$$

□

27. To prove:

$$\sqrt{2}(\sin x-\cos x)\sin(x+45°)=1-2\cos^2 x$$

Proof:

$$\begin{aligned}\text{L.H.S.} &= \sqrt{2}(\sin x-\cos x)\sin(x+45°)\\ &= \sqrt{2}(\sin x-\cos x)\,(\sin x\cos 45°\\ &\qquad +\cos x\sin 45°)\\ &= \sqrt{2}(\sin x-\cos x)\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)\\ &= (\sin x-\cos x)(\sin x+\cos x)\\ &= \sin^2 x-\cos^2 x\\ &= (1-\cos^2 x)-\cos^2 x\\ &= 1-2\cos^2 x\\ &= \text{R.H.S}\end{aligned}$$

□

28. To prove: $\tan\left(\theta+\frac{\pi}{4}\right)=\frac{1+2\sin\theta\cos\theta}{1-2\sin^2\theta}$

Proof:

$$\begin{aligned}\text{L.H.S.} &= \tan\left(\theta+\frac{\pi}{4}\right)\\ &= \frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\tan\pi 4}\\ &= \frac{\tan\theta+1}{1-\tan\theta\times 1}\\ &= \frac{\frac{\sin\theta}{\cos\theta}+1}{1-\frac{\sin\theta}{\cos\theta}}\\ &= \frac{\frac{\sin\theta+\cos\theta}{\cos\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}\\ &= \frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}\\ &= \frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}\times\frac{\sin\theta+\cos\theta}{\cos\theta+\sin\theta}\\ &= \frac{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta}{\cos^2\theta-\sin^2\theta}\\ &= \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{1-\sin^2\theta-\sin^2\theta}\\ &= \frac{1+2\sin\theta\cos\theta}{1-2\sin^2\theta}\\ &= \text{R.H.S}\end{aligned}$$

□

29. $\sin x\cos\frac{\pi}{6}+\cos x\sin\frac{\pi}{6}=\sin(x+\frac{\pi}{6})$ so the equation to solve becomes

$$\sin(x+\frac{\pi}{6})=\frac{1}{\sqrt{2}}$$

This has solutions with $x+\frac{\pi}{6}$ in the 1st and 2nd quadrant.

$$\begin{aligned}x+\frac{\pi}{6} &= \frac{\pi}{4} & x+\frac{\pi}{6} &= \pi-\frac{\pi}{4}\\ x &= \frac{\pi}{4}-\frac{\pi}{6} & x &= \pi-\frac{\pi}{4}-\frac{\pi}{6}\\ x &= \frac{\pi}{12} & x &= \frac{7\pi}{12}\end{aligned}$$

30. $\cos x\cos 20°+\sin x\sin 20°=\cos(x-20°)$ so the equation to solve becomes

$$\cos(x-20°)=\frac{1}{2}$$

This has solutions with $x - 20°$ in the 1st and 4th quadrant.

$$x - 20° = 60° \qquad x - 20° = 360° - 60°$$
$$x = 80° \qquad x - 20° = 300°$$
$$x = 320°$$

31. $\sin x \cos 70° + \cos x \sin 70° = 0.5$
$$\sin(x + 70°) = 0.5$$
This has solutions for $x + 70°$ in the 1st and 2nd quadrant.

$$x + 70° = 30° \qquad x + 70° = 180 - 30°$$
$$x = -40° \qquad x + 70° = 150°$$
$$x = 80°$$

32.
$$\begin{aligned}\sin(x + 30°) &= \cos x\\ \sin x \cos 30° + \cos x \sin 30° &= \cos x\\ \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x &= \cos x\\ \frac{\sqrt{3}}{2}\sin x &= \frac{1}{2}\cos x\\ \sqrt{3}\sin x &= \cos x\\ \sqrt{3}\frac{\sin x}{\cos x} &= 1\\ \tan x &= \frac{1}{\sqrt{3}}\end{aligned}$$
Solutions in 1st and 3rd quadrant.

$$x = 30° \qquad x = 180° + 30°$$
$$= 210°$$

Exercise 3C

1. $\cos \mathrm{A} = -\frac{4}{5}$ (by Pythagoras, and given that A is in the 2nd quadrant) and $\tan \mathrm{A} = -\frac{3}{4}$.

(a) $$\begin{aligned}\sin 2\mathrm{A} &= 2\sin \mathrm{A}\cos \mathrm{A}\\ &= 2 \times \frac{3}{5} \times -\frac{4}{5}\\ &= -\frac{24}{25}\end{aligned}$$

(b) $$\begin{aligned}\cos 2\mathrm{A} &= 2\cos^2 A - 1\\ &= 2 \times \left(\frac{4}{5}\right)^2 - 1\\ &= \frac{7}{25}\end{aligned}$$
(Alternatively, once you've found $\sin 2\mathrm{A}$ use Pythagoras to find $\cos 2\mathrm{A}$.)

(c) $$\begin{aligned}\tan 2\mathrm{A} &= \frac{2\tan \mathrm{A}}{1 - \tan^2 A}\\ &= \frac{2 \times -\frac{3}{4}}{1 - (-\frac{3}{4})^2}\\ &= \frac{-\frac{6}{4}}{\frac{7}{16}}\\ &= -\frac{3}{2} \times \frac{16}{7}\\ &= -\frac{24}{7}\end{aligned}$$

2. Think of the right-angled triangle with sides 5, 12 and 13, but bear in mind that B is in the 3rd quadrant, so we have $\sin \mathrm{B} = -\frac{5}{13}$, $\cos \mathrm{B} = -\frac{12}{13}$.

(a) $$\begin{aligned}\sin 2\mathrm{B} &= 2\sin \mathrm{B}\cos \mathrm{B}\\ &= 2(-\frac{5}{13})(-\frac{12}{13})\\ &= \frac{120}{169}\end{aligned}$$

(b) $$\begin{aligned}\cos 2\mathrm{B} &= 2\cos^2 \mathrm{B} - 1\\ &= 2(-\frac{12}{13})^2 - 1\\ &= \frac{288 - 169}{169}\\ &= \frac{119}{169}\end{aligned}$$

(c) $$\begin{aligned}\tan 2\mathrm{B} &= \frac{\sin 2\mathrm{B}}{\cos 2\mathrm{B}}\\ &= \frac{120}{119}\end{aligned}$$
(We could use the double-angle formula for tangent, but since we have found sine and cosine this is simpler.)

3. (a) $6\sin \mathrm{A}\cos \mathrm{A} = 3(2\sin \mathrm{A}\cos \mathrm{A}) = 3\sin 2\mathrm{A}$

(b) $$\begin{aligned}4\sin 2\mathrm{A}\cos 2\mathrm{A} &= 2(2\sin 2\mathrm{A}\cos 2\mathrm{A})\\ &= 2\sin(2 \times 2\mathrm{A})\\ &= 2\sin 4\mathrm{A}\end{aligned}$$

(c) $$\begin{aligned}\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{A}}{2} &= \frac{1}{2}(2\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{A}}{2})\\ &= \frac{1}{2}\sin(2 \times \frac{\mathrm{A}}{2})\\ &= \frac{1}{2}\sin \mathrm{A}\end{aligned}$$

4. (a)
$$\begin{aligned}2\cos^2 2\text{A} - 2\sin^2 2\text{A} &= 2(\cos^2 2\text{A} - \sin^2 2\text{A})\\ &= 2\cos(2\times 2\text{A})\\ &= 2\cos 4\text{A}\end{aligned}$$

(b)
$$\begin{aligned}1 - 2\sin^2\frac{\text{A}}{2} &= \cos(2\times\frac{\text{A}}{2})\\ &= \cos\text{A}\end{aligned}$$

(c)
$$\begin{aligned}2\cos^2 2\text{A} - 1 &= \cos(2\times 2\text{A})\\ &= \cos 4\text{A}\end{aligned}$$

5. $\sqrt{25^2 - 24^2} = 7$. Think of the right-angled triangle with sides 7, 24 and 25, but bear in mind that θ is obtuse (so in the 2nd quadrant), so we have $\sin\theta = -\frac{5}{13}$, $\cos B = -\frac{12}{13}$.

(a)
$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\times\frac{7}{25}\times -\frac{24}{25}\\ &= -\frac{336}{625}\end{aligned}$$

(b)
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\times\left(\frac{24}{25}\right)^2 - 1\\ &= \frac{1152-625}{625}\\ &= \frac{527}{625}\end{aligned}$$

(c)
$$\begin{aligned}\tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta}\\ &= -\frac{336}{527}\end{aligned}$$

6.
$$\begin{aligned}4\sin x\cos x &= 1\\ 2(2\sin x\cos x) &= 1\\ 2\sin 2x &= 1\\ \sin 2x &= \frac{1}{2}\end{aligned}$$

This will have four solutions with $2x$ in 1st and 2nd quadrant.

$$\begin{aligned}2x &= 30^\circ & \text{or}\quad 2x &= 180^\circ - 30^\circ\\ x &= 15^\circ & 2x &= 150^\circ\\ & & x &= 75^\circ\end{aligned}$$

or

$$\begin{aligned}2x &= 360^\circ + 30^\circ & \text{or}\quad 2x &= 540^\circ - 30^\circ\\ 2x &= 390^\circ & 2x &= 510^\circ\\ x &= 195 & x &= 255^\circ\end{aligned}$$

7.
$$\begin{aligned}\sin 2x + \cos x &= 0\\ 2\sin x\cos x + \cos x &= 0\\ \cos x(2\sin x + 1) &= 0\end{aligned}$$

The Null Factor Law gives:

$$\begin{aligned}\cos x &= 0 & \text{or}\quad 2\sin x + 1 &= 0\\ x &= \pm 90^\circ & 2\sin x &= -1\\ & & \sin x &= -\frac{1}{2}\\ & & x &= -30^\circ\\ & & \text{or } x &= -180^\circ + 30^\circ\\ & & &= -150^\circ\end{aligned}$$

8.
$$\begin{aligned}2\sin 2x - \sin x &= 0\\ 4\sin x\cos x - \sin x &= 0\\ \sin x(4\cos x - 1) &= 0\end{aligned}$$

The Null Factor Law gives:

$$\begin{aligned}\sin x &= 0 & \text{or}\quad 4\cos x - 1 &= 0\\ x &= 0^\circ & 4\cos x &= 1\\ \text{or } x &= 180^\circ & \cos x &= \frac{1}{4}\\ \text{or } x &= 360^\circ & x &= 75.5^\circ\\ & & \text{or } x &= 360^\circ - 75.5^\circ\\ & & &= 284.5^\circ\end{aligned}$$

9.
$$\begin{aligned}2\sin x\cos x &= \cos 2x\\ \sin 2x &= \cos 2x\\ \tan 2x &= 1\end{aligned}$$

This will have 4 solutions (since $\tan 2x$ has a period of $\frac{\pi}{2}$ and we want solutions for $0 \le x \le 2\pi$) in the 1st and 3rd quadrant.

$$\begin{aligned}2x &= \frac{\pi}{4} & \text{or}\quad 2x &= \pi + \frac{\pi}{4}\\ x &= \frac{\pi}{8} & &= \frac{5\pi}{4}\\ & & x &= \frac{5\pi}{8}\end{aligned}$$

$$\begin{aligned}\text{or}\quad 2x &= 2\pi + \frac{\pi}{4} & \text{or}\quad 2x &= 3\pi + \frac{\pi}{4}\\ &= \frac{9\pi}{4} & &= \frac{13\pi}{4}\\ x &= \frac{9\pi}{8} & x &= \frac{13\pi}{8}\end{aligned}$$

10.
$$\begin{aligned}\cos 2x + 1 - \cos x &= 0\\ 2\cos^2 x - 1 + 1 - \cos x &= 0\\ 2\cos^2 x - \cos x &= 0\\ \cos x(2\cos x - 1) &= 0\end{aligned}$$

The Null Factor Law gives:

$$\begin{aligned} \cos x &= 0 & \text{or} \quad 2\cos x - 1 &= 0 \\ x &= \frac{\pi}{2} & 2\cos x &= 1 \\ \text{or } x &= \frac{3\pi}{2} & \cos x &= \frac{1}{2} \\ & & x &= \frac{\pi}{3} \\ & & \text{or } x &= 2\pi - \frac{\pi}{3} \\ & & &= \frac{5\pi}{3} \end{aligned}$$

11.

$$\begin{aligned} \cos 2x + \sin x &= 0 \\ 1 - 2\sin^2 x + \sin x &= 0 \\ 2\sin^2 x - \sin x - 1 &= 0 \\ (2\sin x + 1)(\sin x - 1) &= 0 \end{aligned}$$

The Null Factor Law gives:

$$\begin{aligned} 2\sin x + 1 &= 0 & \text{or} \quad \sin x - 1 &= 0 \\ 2\sin x &= -1 & \sin x &= 1 \\ \sin x &= -\frac{1}{2} & x &= \frac{\pi}{4} \\ x &= -\frac{\pi}{6} \\ \text{or} \quad x &= -\pi + \frac{\pi}{6} \\ &= -\frac{5\pi}{6} \end{aligned}$$

12. We can easily express $\sin^2 x$ and $\cos 2x$ in terms of either sine or cosine but $\cos x$ is not as readily changed to sine, so we'll go with cosine. (In this case it turns out not to really matter, but it's still a good principle.)

$$\begin{aligned} 2\sin^2 x + 5\cos x + \cos 2x &= 3 \\ 2(1 - \cos^2 x) + 5\cos x + (2\cos^2 x - 1) &= 3 \\ 2 - 2\cos^2 x + 5\cos x + 2\cos^2 x - 1 &= 3 \\ 1 + 5\cos x &= 3 \\ 5\cos x &= 2 \\ \cos x &= 0.4 \end{aligned}$$

The display given provides the prime (first quadrant) solution to this:

$$x = 66.4^\circ$$

There will also be a solution in the 4th quadrant:

$$x = 360 - 66.4 = 293.6^\circ$$

As we continue up to 540° we pass through the 1st quadrant again and find the third solution:

$$x = 360 + 66.4 = 426.4^\circ$$

13. L.H.S.:

$$\begin{aligned} \sin 2\theta \tan\theta &= 2\sin\theta\cos\theta \times \frac{\sin\theta}{\cos\theta} \\ &= 2\sin^2\theta \\ &= \text{R.H.S.} \end{aligned}$$

□

14. L.H.S.:

$$\begin{aligned} \cos\theta\sin 2\theta &= \cos\theta \times 2\sin\theta\cos\theta \\ &= 2\sin\theta\cos^2\theta \\ &= 2\sin\theta(1 - \sin^2\theta) \\ &= 2\sin\theta - 2\sin^3\theta) \\ &= \text{R.H.S.} \end{aligned}$$

□

15. L.H.S.:

$$\begin{aligned} \frac{1 - \cos 2\theta}{1 + \cos 2\theta} &= \frac{1 - (1 - 2\sin^2\theta)}{1 + (2\cos^2\theta - 1)} \\ &= \frac{1 - 1 + 2\sin^2\theta}{1 + 2\cos^2\theta - 1} \\ &= \frac{2\sin^2\theta}{2\cos^2\theta} \\ &= \frac{\sin^2\theta}{\cos^2\theta} \\ &= \tan^2\theta \\ &= \text{R.H.S.} \end{aligned}$$

□

16. L.H.S.:

$$\begin{aligned} \sin\theta\tan\frac{\theta}{2} &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \times \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\ &= 2\sin^2\frac{\theta}{2} \\ &= 2(1 - \cos^2\frac{\theta}{2}) \\ &= 2 - 2\cos^2\frac{\theta}{2} \\ &= \text{R.H.S.} \end{aligned}$$

□

17. Although the R.H.S. looks more complicated, it will probably be easier to expand the left hand side than work out how to get a multiple angle function out of the right hand side. L.H.S.:

$$\begin{aligned} \sin 4\theta &= 2\sin 2\theta\cos 2\theta \\ &= 2(2\sin\theta\cos\theta)(\cos^2\theta - \sin^2\theta) \\ &= (4\sin\theta\cos\theta)(\cos^2\theta - \sin^2\theta) \\ &= 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta \\ &= \text{R.H.S.} \end{aligned}$$

□

18. L.H.S.:

$$\begin{aligned}\frac{\sin 2\theta - \sin\theta}{1-\cos\theta+\cos 2\theta} &= \frac{2\sin\theta\cos\theta - \sin\theta}{1-\cos\theta+(2\cos^2\theta-1)}\\ &= \frac{\sin\theta(2\cos\theta-1)}{2\cos^2\theta-\cos\theta}\\ &= \frac{\sin\theta(2\cos\theta-1)}{\cos\theta(2\cos\theta-1)}\\ &= \frac{\sin\theta}{\cos\theta}\\ &= \tan\theta\\ &= \text{R.H.S.}\end{aligned}$$

□

19. L.H.S.:

$$\begin{aligned}\cos 4\theta &= 2\cos^2 2\theta - 1\\ &= 2(\cos 2\theta)^2 - 1\\ &= 2(2\cos^2\theta-1)^2 - 1\\ &= 2(2\cos^2\theta-1)(2\cos^2\theta-1) - 1\\ &= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1\\ &= 8\cos^4\theta - 8\cos^2\theta + 2 - 1\\ &= 8\cos^4\theta - 8\cos^2\theta + 1\\ &= 1 - 8\cos^2\theta + 8\cos^4\theta\\ &= \text{R.H.S.}\end{aligned}$$

□

Exercise 3D

1. To obtain the form $a\cos(\theta+\alpha)$ we need to rearrange our expression so it looks like the expansion of this:

$$a\cos(\theta+\alpha) = a(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

$$\begin{aligned}\sqrt{3^2+4^2} &= 5\\ 3\cos\theta - 4\sin\theta &= 5\left(\frac{3}{5}\cos\theta - \frac{4}{5}\sin\theta\right)\\ &= 5(\cos\theta\cos\alpha - \sin\theta\sin\alpha\end{aligned}$$

giving $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$:

$$\alpha = \cos^{-1}\frac{3}{5} = 53.1°$$

hence

$$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1°)$$

2.
$$\begin{aligned}\sqrt{12^2+5^2} &= 13\\ 12\cos\theta - 5\sin\theta &= 13\left(\frac{12}{13}\cos\theta - \frac{5}{13}\sin\theta\right)\\ &= 13(\cos\theta\cos\alpha - \sin\theta\sin\alpha\end{aligned}$$

giving $\cos\alpha = \frac{12}{13}$ and $\sin\alpha = \frac{5}{13}$:

$$\alpha = \cos^{-1}\frac{12}{13} = 22.6°$$

hence

$$12\cos\theta - 5\sin\theta = 13\cos(\theta + 22.6°)$$

3. To obtain the form $a\cos(\theta-\alpha)$ we need to rearrange our expression so it looks like the expansion of this:

$$a\cos(\theta-\alpha) = a(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

$$\begin{aligned}\sqrt{4^2+3^2} &= 5\\ 4\cos\theta + 3\sin\theta &= 5\left(\frac{4}{5}\cos\theta + \frac{3}{5}\sin\theta\right)\\ &= 5(\cos\theta\cos\alpha + \sin\theta\sin\alpha)\end{aligned}$$

giving $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$:

$$\alpha = \cos^{-1}\frac{4}{5} = 0.64$$

hence

$$4\cos\theta + 3\sin\theta = 5\cos(\theta - 0.64)$$

4.
$$\begin{aligned}\sqrt{7^2+24^2} &= 25\\ 7\cos\theta + 24\sin\theta &= 25\left(\frac{7}{25}\cos\theta + \frac{24}{25}\sin\theta\right)\\ &= 25(\cos\theta\cos\alpha + \sin\theta\sin\alpha)\end{aligned}$$

giving $\cos\alpha = \frac{7}{25}$ and $\sin\alpha = \frac{24}{25}$:

$$\alpha = \cos^{-1}\frac{7}{25} = 1.29$$

hence

$$7\cos\theta + 24\sin\theta = 25\cos(\theta - 1.29)$$

5. To obtain the form $a\sin(\theta+\alpha)$ we need to rearrange our expression so it looks like the expansion of this:

$$a\sin(\theta+\alpha) = a(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$

$$\sqrt{5^2+12^2} = 13$$
$$5\sin\theta + 12\cos\theta = 13\left(\frac{5}{13}\sin\theta + \frac{12}{13}\cos\theta\right) = 13(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
giving $\cos\alpha = \frac{5}{13}$ and $\sin\alpha = \frac{12}{13}$:
$$\alpha = \cos^{-1}\frac{5}{13} = 67.4°$$
hence
$$5\sin\theta + 12\cos\theta = 13\sin(\theta + 67.4°)$$

6. $$\sqrt{7^2+24^2} = 25$$
$$7\sin\theta + 24\cos\theta = 25\left(\frac{7}{25}\sin\theta + \frac{24}{25}\cos\theta\right) = 25(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
giving $\cos\alpha = \frac{7}{25}$ and $\sin\alpha = \frac{24}{25}$:
$$\alpha = \cos^{-1}\frac{7}{25} = 73.7°$$
hence
$$7\sin\theta + 24\cos\theta = 25\sin(\theta + 73.7°)$$

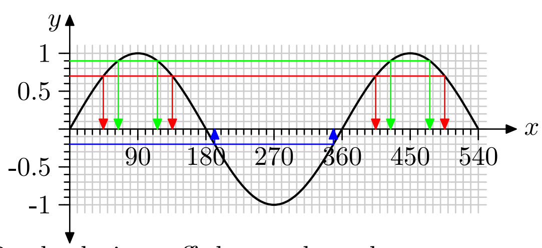
7. To obtain the form $a\sin(\theta-\alpha)$ we need to rearrange our expression so it looks like the expansion of this:
$$a\sin(\theta-\alpha) = a(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$
$$\sqrt{4^2+3^2} = 5$$
$$4\sin\theta - 3\cos\theta = 5\left(\frac{4}{5}\sin\theta - \frac{3}{5}\cos\theta\right) = 5(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$
giving $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$:
$$\alpha = \cos^{-1}\frac{4}{5} = 0.64$$
hence
$$4\sin\theta - 3\cos\theta = 5\sin(\theta - 0.64)$$

8. $$\sqrt{2^2+3^2} = \sqrt{13}$$
$$2\sin\theta - 3\cos\theta = \sqrt{13}\left(\frac{2}{\sqrt{13}}\sin\theta - \frac{3}{\sqrt{13}}\cos\theta\right) = \sqrt{13}(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$
giving $\cos\alpha = \frac{2}{\sqrt{13}}$ and $\sin\alpha = \frac{3}{\sqrt{13}}$:
$$\alpha = \cos^{-1}\frac{2}{\sqrt{13}} = 0.98$$
hence
$$2\sin\theta - 3\cos\theta = \sqrt{13}\sin(\theta - 0.98)$$

9. (a) $$\sqrt{1^2+1^2} = \sqrt{2}$$
$$\cos\theta + \sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right) = \sqrt{2}(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$
giving $\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$:
$$\alpha = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$
hence
$$\cos\theta + \sin\theta = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$$

(b) The maximum value of $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$ is $\sqrt{2}$ (its amplitude) and occurs when
$$\cos\left(\theta - \frac{\pi}{4}\right) = 1$$
$$\theta - \frac{\pi}{4} = 0$$
$$\theta = \frac{\pi}{4}$$

Miscellaneous Exercise 3

1. 

Read solutions off the graph as shown.

2. Read the answer for (a) directly from the graph. For (b), read the period directly from the graph. Amplitude$=\frac{4--2}{2} = 3$.

3.
$$\begin{aligned}
|\mathbf{a}| &= |\mathbf{b}| \\
1^2 + p^2 &= 5^2 + 5^2 \\
1 + p^2 &= 50 \\
p^2 &= 49 \\
p &= \pm 7 \\
\mathbf{c} + (2\mathbf{i} - 3\mathbf{j}) &= \mathbf{a} \\
\mathbf{c} &= (\mathbf{i} \pm 7\mathbf{j}) - (2\mathbf{i} - 3\mathbf{j}) \\
\mathbf{c} &= (1-2)\mathbf{i} + (\pm 7 + 3)\mathbf{j} \\
\mathbf{c} &= -\mathbf{i} + 10\mathbf{j} \\
\text{or} \quad \mathbf{c} &= -\mathbf{i} - 4\mathbf{j}
\end{aligned}$$

4. $\sin x° = -0.53$ will have solutions in the 3rd and 4th quadrants: two solutions for each full cycle of the sine graph. Thus in the interval $-360 \le x \le 360$ we will have four solutions:

- $x = -32$ (4th quadrant)
- $x = -180 + 32 = -148$ (3rd quadrant)
- $x = 180 + 32 = 212$ (3rd quadrant)
- $x = 360 - 32 = 328$ (4th quadrant)

5. Beginning with the Left Hand Side as most complicated (even though it is shorter):

$$\begin{aligned}
\cos 3\theta &= \cos(2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta \\
&= 2\cos^3\theta - \cos\theta - (2 - 2\cos^2\theta)\cos\theta \\
&= 2\cos^3\theta - \cos\theta - (2\cos\theta - 2\cos^3\theta) \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta \\
&= \text{Right Hand Side}
\end{aligned}$$

□

6.
$$\begin{aligned}
\frac{1}{z} &= \frac{2+7\text{i}}{1-\text{i}} \\
z &= \frac{1-\text{i}}{2+7\text{i}} \\
&= \frac{1-\text{i}}{2+7\text{i}} \times \frac{2-7\text{i}}{2-7\text{i}}
\end{aligned}$$

7.
$$\begin{aligned}
(a+b\text{i})^2 &= 5 - 12\text{i} \\
a^2 + 2ab\text{i} - b^2 &= 5 - 12\text{i} \\
a^2 - b^2 + 2ab\text{i} &= 5 - 12\text{i} \\
2ab &= -12 \\
b &= -\frac{6}{a} \\
a^2 - b^2 &= 5 \\
a^2 - (-\frac{6}{a})^2 &= 5 \\
a^2 - \frac{36}{a^2} &= 5 \\
a^4 - 36 &= 5a^2 \\
a^4 - 5a^2 - 36 &= 0 \\
(a^2 - 9)(a^2 + 4) &= 0
\end{aligned}$$

If we now apply the null factor law we should be able to see that $a^2 + 4 = 0$ has no real solutions, so

$$\begin{aligned}
a^2 - 9 &= 0 \\
a^2 &= 9 \\
a &= \pm 3
\end{aligned}$$

Now substitute back to find b:

$$\begin{aligned}
b &= -\frac{6}{a} \\
&= -\frac{6}{\pm 3} \\
&= -(\pm 2) \\
&= \mp 2
\end{aligned}$$

So we have either $a = 3$, $b = -2$ or $a = -3$, $b = 2$. (Given the square in the original problem we should have expected to get a pair of solutions like this where one is the opposite of the other.)

8. (a)
$$\begin{aligned}
p + q &= 2 - 3\text{i} + 5 + \text{i} \\
&= 7 - 2\text{i}
\end{aligned}$$

(b)
$$\begin{aligned}
p \div q &= \frac{2-3\text{i}}{5+\text{i}} \\
&= \frac{(2-3\text{i})(5-\text{i})}{(5+\text{i})(5-\text{i})} \\
&= \frac{10 - 2\text{i} - 15\text{i} - 3}{25+1} \\
&= \frac{7-17\text{i}}{26} \\
&= \frac{7}{26} - \frac{17}{26}\text{i}
\end{aligned}$$

9. Given $z = a + b\text{i}$ then $\bar{z} = a - b\text{i}$.

$$\begin{aligned}
z + 2\bar{z} &= 9 + 5\text{i} \\
a + b\text{i} + 2(a - b\text{i}) &= 9 + 5\text{i} \\
a + b\text{i} + 2a - 2b\text{i} &= 9 + 5\text{i} \\
3a - b\text{i} &= 9 + 5\text{i} \\
a &= 3 \\
b &= -5
\end{aligned}$$

10. (a)
$$\begin{aligned}2x^3-5x^2+8x-3&=(px-q)(x^2+rx+3)\\&=px^3+prx^2+3px-qx^2-qrx-3q\\&=px^3+(pr-q)x^2+(3p-qr)x-3q\end{aligned}$$

Equating like terms:

$$\begin{aligned}2x^3&=px^3\\p&=2\\-3&=-3q\\q&=1\\-5x^2&=(pr-q)x^2\\-5&=pr-q\\&=2r-1\\2r-1&=-5\\2r&=-4\\r&=-2\end{aligned}$$

check: $8x=(3p-qr)x$

$$\begin{aligned}3p-qr&=3\times2-1\times-2\\&=8\end{aligned}$$

(b) Substitute p,q,r to obtain a factorization:

$2x^3-5x^2+8x-3=(2x-1)(x^2-2x+3)$

then use the null factor law:

$$\begin{aligned}2x-1&=0\\x&=\frac{1}{2}\end{aligned}$$

or $x^2-2x+3=0$

$$\begin{aligned}x&=\frac{2\pm\sqrt{(-2)^2-4\times1\times3}}{2\times1}\\&=\frac{2\pm\sqrt{-8}}{2}\\&=\frac{2\pm\sqrt{8}\mathrm{i}}{2}\\&=\frac{2\pm2\sqrt{2}\mathrm{i}}{2}\\x&=1+\sqrt{2}\mathrm{i}\\\text{or } x&=1-\sqrt{2}\mathrm{i}\end{aligned}$$

11. In the 1st quadrant we obtain

$$\begin{aligned}x+\frac{\pi}{4}&=\frac{\pi}{3}\\x&=\frac{\pi}{3}-\frac{\pi}{4}\\&=\frac{\pi}{12}\end{aligned}$$

In the 2nd quadrant:

$$\begin{aligned}x+\frac{\pi}{4}&=\frac{2\pi}{3}\\x&=\frac{2\pi}{3}-\frac{\pi}{4}\\&=\frac{8\pi-3\pi}{12}\\&=\frac{5\pi}{12}\end{aligned}$$

12.
$$\begin{aligned}k\sin2\theta&=2\sin^2\theta\\2k\sin\theta\cos\theta&=2\sin^2\theta\\2k\sin\theta\cos\theta-2\sin^2\theta&=0\\2k\sin\theta\cos\theta-2\sin^2\theta&=0\\2\sin\theta(k\cos\theta-\sin\theta)&=0\\\sin\theta(k\cos\theta-\sin\theta)&=0\end{aligned}$$

$\sin\theta=0$ or $k\cos\theta-\sin\theta=0$

$\theta=0$ $\quad k\cos\theta=\sin\theta$

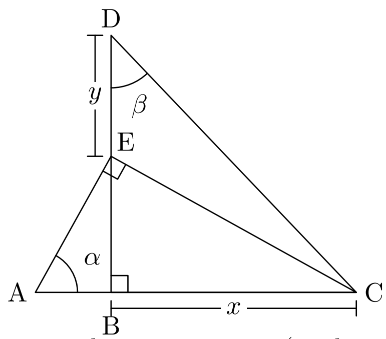
or $\theta=\pi$ $\quad \sin\theta=k\cos\theta$

or $\theta=2\pi$ $\quad \tan\theta=k$

$\theta=p$

or $\theta=\pi+p$

13.



∠AEB is complementary to α (angle sum of a triangle).

∠BEC is complementary to ∠AEB

$\therefore$ ∠BEC $\cong\alpha$

$$\begin{aligned}\tan\alpha&=\frac{x}{\mathrm{EB}}\\\mathrm{EB}&=\frac{x}{\tan\alpha}\\&=\frac{x\cos\alpha}{\sin\alpha}\\\tan\beta&=\frac{x}{y+\mathrm{EB}}\\y+\mathrm{EB}&=\frac{x}{\tan\beta}\\y&=\frac{x}{\tan\beta}-\mathrm{EB}\\y&=\frac{x\cos\beta}{\sin\beta}-\frac{x\cos\alpha}{\sin\alpha}\\&=\frac{x\sin\alpha\cos\beta}{\sin\alpha\sin\beta}-\frac{x\cos\alpha\sin\beta}{\sin\alpha\sin\beta}\\&=\frac{x\sin\alpha\cos\beta-x\cos\alpha\sin\beta}{\sin\alpha\sin\beta}\\&=\frac{x(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{\sin\alpha\sin\beta}\\&=\frac{x\sin(\alpha-\beta)}{\sin\alpha\sin\beta}\end{aligned}$$

□