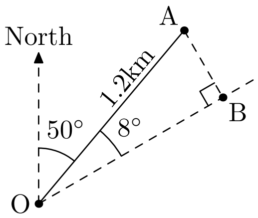
Chapter 8

Exercise 8A

1. Let O be the starting point, A be the first checkpoint and B be the point along the competitor's path that is nearest A.

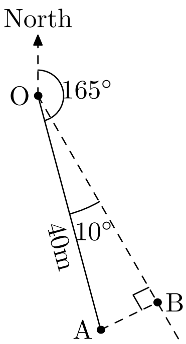


To be the nearest point, B must be such that AB is perpendicular to OB. This means that OAB is a right angled triangle.

$$\begin{aligned} \sin 8^\circ &= \frac{AB}{OA} \\ &= \frac{AB}{1.2} \\ AB &= 1.2\sin 8^\circ \\ &= 0.167\text{km} \end{aligned}$$

The competitor comes within about 170m of the checkpoint.

2. Let O be the position of the batsman, A be the position of the fielder and B be the point along the ball's path that is nearest A.

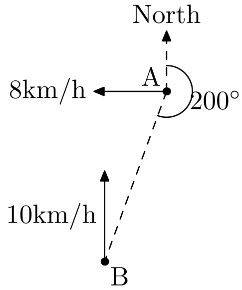


To be the nearest point, B must be such that AB is perpendicular to OB. This means that OAB is a right angled triangle.

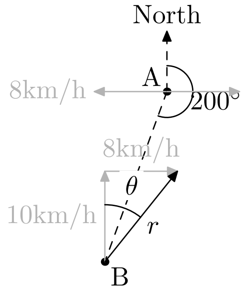
$$\begin{aligned} \sin 10^\circ &= \frac{AB}{OA} \\ &= \frac{AB}{40} \\ AB &= 40\sin 10^\circ \\ &= 6.945\text{m} \end{aligned}$$

The ball passes about 6.9m from the fielder who can not be confident of stopping it.

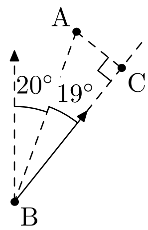
3. Impose a velocity of 8km/h due east to consider the situation from the point of view of an observer on vessel A.



becomes



$$\begin{aligned} \tan\theta &= \frac{8}{10} \\ \theta &= \tan^{-1} 0.8 \\ &= 39^\circ \end{aligned}$$

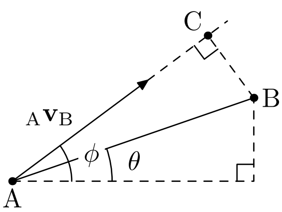


$$\begin{aligned} \sin 19^\circ &= \frac{AC}{AB} \\ AC &= AB\sin 19^\circ \\ &= 2\sin 19^\circ \\ &= 0.64\text{km} \end{aligned}$$

The vessels pass within about 640m of each other.

4. First with vectors:

$$\begin{aligned} \overrightarrow{\text{AB}} &= \mathbf{r}_\text{B} - \mathbf{r}_\text{A} \\ &= (22\mathbf{i} + 21\mathbf{j}) - (-10\mathbf{i} + 10\mathbf{j}) \\ &= (32\mathbf{i} + 11\mathbf{j})\text{km} \\ {}_\text{A}\mathbf{v}_\text{B} &= \mathbf{v}_\text{A} - \mathbf{v}_\text{B} \\ &= (15\mathbf{i} + 10\mathbf{j}) - (-5\mathbf{i}) \\ &= (20\mathbf{i} + 10\mathbf{j})\text{km/h} \end{aligned}$$



$$\begin{aligned}
\tan\theta &= \frac{11}{32}\\
\theta &= \tan^{-1}\frac{11}{32}\\
\tan\phi &= \frac{10}{20}\\
\phi &= \tan^{-1}\frac{1}{2}\\
\sin(\phi-\theta) &= \frac{\text{BC}}{\text{BA}}\\
\text{BC} &= \text{BA}\sin(\phi-\theta)\\
&= \sqrt{32^2+11^2}\sin(\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{11}{32})\\
&= 4.47\text{km}\\
\text{AC} &= \sqrt{32^2+11^2}\cos(\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{11}{32})\\
&= 33.54\text{km}\\
t &= \frac{33.54}{|_{\text{A}}\mathbf{v}_{\text{B}}}\\
&= \frac{33.54}{\sqrt{20^2+10^2}}\\
&= 1.5\text{hours}
\end{aligned}$$

The ships pass within about 4.5km of each other after 1.5 hours, i.e. at 11:30am.

Now using a different approach:

$$\begin{aligned}
\mathbf{r}_{\text{A}}(t) &= (-10\mathbf{i}+10\mathbf{j})+t(15\mathbf{i}+10\mathbf{j})\\
&= (-10+15t)\mathbf{i}+(10+10t)\mathbf{j}\\
\mathbf{r}_{\text{B}}(t) &= (22\mathbf{i}+21\mathbf{j})+t(-5\mathbf{i})\\
&= (22-5t)\mathbf{i}+21\mathbf{j}
\end{aligned}$$

let d be the distance AB

$$\begin{aligned}
d(t) &= |\mathbf{r}_{\text{B}}(t)-\mathbf{r}_{\text{A}}(t)|\\
&= |[(22-5t)\mathbf{i}+21\mathbf{j}]\\
&\quad -[(-10+15t)\mathbf{i}+(10+10t)\mathbf{j}]\\
&= |(32-20t)\mathbf{i}+(11-10t)\mathbf{j}|\\
d^2(t) &= (32-20t)^2+(11-10t)^2\\
&= 1024-1280t+400t^2\\
&\quad +121-220t+100t^2\\
&= 500t^2-1500t+1145
\end{aligned}$$

This has a minimum when $t=\frac{-b}{2a}$, i.e.

$$\begin{aligned}
t &= \frac{1500}{2\times 500}\\
&= 1.5\\
d^2(1.5) &= 500(1.5)^2-1500(1.5)+1145\\
&= 1125-2250+1145\\
&= 20\\
d &= \sqrt{20}\\
&= 2\sqrt{5}\\
&\approx 4.47\text{km}
\end{aligned}$$

For finding the minimum distance there is not much between the two methods, but if you also need to find the time, the quadratic approach may be a little simpler.

5. Using the quadratic approach:

$$\begin{aligned}
\mathbf{r}_{\text{A}}(t) &= 10t\mathbf{i}+(20+10t)\mathbf{j}\\
\mathbf{r}_{\text{B}}(t) &= (24+t)\mathbf{i}+(18+7t)\mathbf{j}\\
\overrightarrow{\text{AB}}(t) &= (24+t)\mathbf{i}+(18+7t)\mathbf{j}\\
&\quad -[10t\mathbf{i}+(20+10t)\mathbf{j}]\\
&= (24-9t)\mathbf{i}+(-2-3t)\mathbf{j}\\
d(t) &= |\overrightarrow{\text{AB}}(t)|\\
&= |(24-9t)\mathbf{i}+(-2-3t)\mathbf{j}|\\
d^2(t) &= (24-9t)^2+(-2-3t)^2\\
&= 576-432t+81t^2\\
&\quad +4+12t+9t^2\\
&= 90t^2-420t+580
\end{aligned}$$

minimum is where

$$\begin{aligned}
t &= \frac{420}{2\times 90}\\
&= \frac{7}{3}\\
d\left(\frac{7}{3}\right) &= \sqrt{90\left(\frac{7}{3}\right)^2-420\left(\frac{7}{3}\right)+580}\\
&= 3\sqrt{10}
\end{aligned}$$

The minimum distance will be $3\sqrt{10}\approx 9.5$km. (This will occur after 2 hours and 20 minutes, i.e. at 2:20pm.)

6. The position vectors of A and B at time t are:

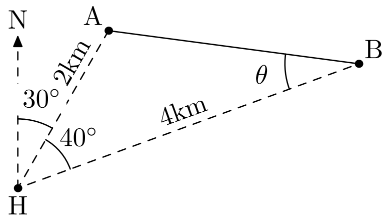
$$\begin{aligned}
\mathbf{r}_{\text{A}} &= (11-5t)\mathbf{i}+(5+3t)\mathbf{j}\\
\mathbf{r}_{\text{B}} &= (-7.5+3t)\mathbf{i}+(9.5-3t)\mathbf{j}\\
\overrightarrow{\text{AB}}(t) &= (-18.5+8t)\mathbf{i}+(4.5-6t)\mathbf{j}\\
d^2(t) &= (-18.5+8t)^2+(4.5-6t)^2\\
&= 100t^2-350t+362.5
\end{aligned}$$

minimum where

$$\begin{aligned}
t &= \frac{350}{2\times 200}\\
&= 1.75\text{s}\\
d(1.75) &= \sqrt{100(1.75)^2-350(1.75)+362.5}\\
&= 7.5\text{m}
\end{aligned}$$

The least distance is 7.5m and occurs at $t=1.75$ seconds.

7. First find $\overrightarrow{\text{AB}}$:



Find the length AB using the cosine rule:

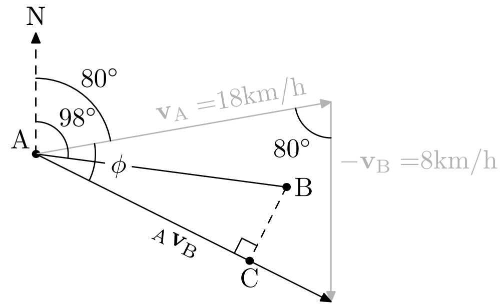
$$\begin{aligned}\text{AB} &= \sqrt{4^2 + 2^2 - 2 \times 4 \times 2 \cos 40^\circ} \\ &= 2.783\text{km}\end{aligned}$$

Find an angle using the sine rule. The angle θ is unambiguous (it can't be obtuse because it's not opposite the longest side of the triangle).

$$\begin{aligned}\frac{\sin\theta}{2} &= \frac{\sin 40^\circ}{2.783} \\ \theta &= \sin^{-1}\frac{2\sin 40^\circ}{2.783} \\ &= 27.51^\circ\end{aligned}$$

so the bearing from A to B is 70° + 27.51 = 097.51° (or 098° to the nearest degree).

The velocity of B relative to A:



Using the cosine rule

$$\begin{aligned}|{}_\text{A}\mathbf{v}_\text{B}| &= \sqrt{8^2 + 18^2 - 2 \times 8 \times 18 \cos 80^\circ} \\ &= 18.38\end{aligned}$$

Using the sine rule

$$\begin{aligned}\frac{\sin\phi}{8} &= \frac{\sin 80^\circ}{18.38} \\ \phi &= \sin^{-1}\frac{8\sin 80^\circ}{18.38} \\ &= 25.37^\circ \\ \angle\text{BAC} &= \phi - 18 \\ &= 7.37^\circ \\ \sin\angle 7.37^\circ &= \frac{\text{BC}}{\text{AB}} \\ \text{BC} &= \text{AB}\sin\angle 7.37^\circ \\ &= 2.783\sin\angle 7.37^\circ \\ &= 351\text{km} \\ &\approx 400\text{m}\end{aligned}$$

Exercise 8B

1. $\mathbf{a}\cdot\mathbf{b} = 5 \times 3\cos 30^\circ = \frac{15\sqrt{3}}{2}$
2. $\mathbf{a}$ and $\mathbf{c}$ are perpendicular so $\mathbf{a}\cdot\mathbf{c} = 0$
3. $\mathbf{a}\cdot\mathbf{d} = 3 \times 2 \times \cos 180^\circ = -6$
4. $\mathbf{b}\cdot\mathbf{c} = 5 \times 4\cos 60^\circ = 10$
5. $\mathbf{b}\cdot\mathbf{d} = 5 \times 2\cos 150^\circ = -5\sqrt{3}$
6. $\mathbf{c}$ and $\mathbf{d}$ are perpendicular so $\mathbf{c}\cdot\mathbf{d} = 0$
7. $\mathbf{e}$ and $\mathbf{f}$ are perpendicular so $\mathbf{e}\cdot\mathbf{f} = 0$
8. $\mathbf{f}\cdot\mathbf{e} = \mathbf{e}\cdot\mathbf{f} = 0$
9. $\mathbf{e}\cdot\mathbf{e} = e^2 = 2^2 = 4$
10. $\mathbf{f}\cdot\mathbf{f} = f^2 = 3^2 = 9$
11. $\mathbf{f}\cdot\mathbf{g} = 4 \times 3\cos 45^\circ = 6\sqrt{2}$
12. $\mathbf{g}\cdot\mathbf{f} = \mathbf{f}\cdot\mathbf{g} = 6\sqrt{2}$
13. $\mathbf{a}\cdot\mathbf{b} = 4 \times 6\cos 60^\circ = 12$
14. $\mathbf{a}\cdot\mathbf{b} = 8 \times 6\cos 120^\circ = -24$
15. If we draw the vectors tail to tail the angle is 60° so $\mathbf{a}\cdot\mathbf{b} = 2 \times 3\cos 60^\circ = 3$
16. The vectors are perpendicular so the scalar product is 0.
17. $\mathbf{a}\cdot\mathbf{b} = 7 \times 10\cos 150^\circ = -35\sqrt{3}$
18. $\mathbf{a}\cdot\mathbf{b} = 10 \times 20\cos 135^\circ = -100\sqrt{2}$
19. (a) The magnitude of a vector is a scalar.
    (b) The dot product of two vectors is a scalar.
    (c) The sum of two vectors is a vector.
    (d) The difference of two vectors is a vector.

(e) $2\mathbf{b}$ is a vector so this is the sum of two vectors: a vector.

(f) This is the dot product of two vectors: it's a scalar.

(g) $(\mathbf{a}+\mathbf{b})$ is a vector, as is $(\mathbf{c}+\mathbf{d})$, so this is the dot product of two vectors: it's a scalar.

(h) Magnitude is always a scalar.

(i) $\lambda\mathbf{b}$ is a vector so this is the sum of two vectors: a vector.

(j) This is the dot product of two vectors: it's a scalar.

20. (a) $\mathbf{i}\cdot\mathbf{i} = |\mathbf{i}|^2 = 1^2 = 1$

(b) $\mathbf{i}$ and $\mathbf{j}$ are perpendicular so $\mathbf{i}\cdot\mathbf{j} = 0$

(c) $\mathbf{j}\cdot\mathbf{j} = |\mathbf{j}|^2 = 1^2 = 1$

21. (a)
$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}\\ &= a^2-b^2\end{aligned}$$

(b)
$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}+2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}\\ &= a^2+2\mathbf{a}\cdot\mathbf{b}+b^2\end{aligned}$$

(c)
$$\begin{aligned}&(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}-2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}\\ &= a^2-2\mathbf{a}\cdot\mathbf{b}+b^2\end{aligned}$$

(d)
$$\begin{aligned}&(2\mathbf{a}+\mathbf{b})\cdot(2\mathbf{a}-\mathbf{b})\\ &= 2\mathbf{a}\cdot2\mathbf{a}-2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot2\mathbf{a}-\mathbf{b}\cdot\mathbf{b}\\ &= 4\mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}\\ &= 4a^2-b^2\end{aligned}$$

(e)
$$\begin{aligned}&(\mathbf{a}+3\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot2\mathbf{b}+3\mathbf{b}\cdot\mathbf{a}-3\mathbf{b}\cdot2\mathbf{b}\\ &- \mathbf{a}\cdot\mathbf{a}-2\mathbf{a}\cdot\mathbf{b}+3\mathbf{b}\cdot\mathbf{a}-6\mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}-6\mathbf{b}\cdot\mathbf{b}\\ &= a^2+\mathbf{a}\cdot\mathbf{b}-6b^2\end{aligned}$$

(f)
$$\begin{aligned}&\mathbf{a}\cdot(\mathbf{a}-\mathbf{b})+\mathbf{a}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}\\ &= a^2\end{aligned}$$

22. L.H.S.:

$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}-2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-2\mathbf{b}\cdot\mathbf{b}\\ &= a^2-0+0-2b^2\\ &= a^2-2b^2\\ &= \text{R.H.S.}\end{aligned}$$

□

23. (a) This is not true. Non-zero vectors can't be both equal and perpendicular.

(b) This is true. If vectors are perpendicular they have a zero dot product.

(c) This is not (necessarily) true. $ab = 0$ implies one (at least) of the vectors is the zero vector.

(d) This is true. $\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b} = a^2+0$.

24. (a) True. The vectors are perpendicular so their dot product is zero.

(b) True. This follows from (a).

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) &= 0\\ \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c} &= 0\\ \mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot\mathbf{c}\end{aligned}$$

(c) Not true. In fact if this were true $(\mathbf{b}-\mathbf{c})$ would be the zero vector and it doesn't make much sense to talk about perpendicularity with the zero vector. There's an important point here: you can't simplify an expression by 'dividing' by a vector. Division by a vector is undefined. Unlike a scalar multiplication, you can not 'undo' a dot product by any inverse operation.

(d) True. If $\mathbf{a}$ is perpendicular with $(\mathbf{b}-\mathbf{c})$ it must also be perpendicular with its opposite.

25.
$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (x_1\mathbf{i}+y_1\mathbf{j})\cdot(x_2\mathbf{i}+y_2\mathbf{j})\\ &= x_1x_2\mathbf{i}\cdot\mathbf{i}+x_1y_2\mathbf{i}\cdot\mathbf{j}+x_2y_1\mathbf{i}\cdot\mathbf{j}+y_1y_2\mathbf{j}\cdot\mathbf{j}\\ &= x_1x_2(1)+x_1y_2(0)+x_2y_1(0)+y_1y_2(1)\\ &= x_1x_2+y_1y_2\end{aligned}$$

26. (a)
$$\begin{aligned}\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}\\ &= a^2-0\\ &\neq 0\end{aligned}$$

Not true.

(b) Refer to question 21(a) for the first step.

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= a^2-b^2\\ &\neq 0 \text{ unless } a=b\end{aligned}$$

Not necessarily true.

(c) Refer to question 21(b) for the first step.

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= a^2+2\mathbf{a}\cdot\mathbf{b}+b^2\\ &= a^2+0+b^2\end{aligned}$$

True.

27. (a)
$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta\\ 7 &= 5\times3\cos\theta\\ \cos\theta &= \frac{7}{15}\\ \theta &= \cos^{-1}\frac{7}{15}\\ &= 62^\circ\end{aligned}$$

(b) $\mathbf{a}\cdot\mathbf{a} = a^2 = 5^2 = 25$

(c) $\mathbf{b}\cdot\mathbf{b} = b^2 = 3^2 = 9$

(d) Refer question 21(c) for the first step.

$$\begin{aligned}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= a^2 - 2\mathbf{a}\cdot\mathbf{b} + b^2\\ &= 5^2 - 2(7) + 3^2\\ &= 20\end{aligned}$$

(e) You may be tempted to use trigonometry and the angle between vectors, but there's a simpler approach. The dot product of any vector with itself is the square of its magnitude, so the magnitude must be the square root of the dot product:

$$\begin{aligned}|\mathbf{a}-\mathbf{b}| &= \sqrt{(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})}\\ &= \sqrt{20}\\ &= 2\sqrt{5}\end{aligned}$$

28.
$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= (3\mathbf{a}+2\mathbf{b})\cdot(4\mathbf{a}-\mathbf{b})\\ &= 12a^2 - 3\mathbf{a}\cdot\mathbf{b} + 8\mathbf{a}\cdot\mathbf{b} - 2b^2\\ &= 12(3^2) - 3(0) + 8(0) - 2(2^2)\\ &= 108 - 8\\ &= 100\end{aligned}$$

29. The scalar product is, by definition, the product of two vectors. $(\mathbf{b}+\mathbf{c})$ and $(\mathbf{b}-\mathbf{c})$ are both vectors so $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$ and $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$ but $(\mathbf{b}\cdot\mathbf{c})$ is a scalar, so $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$ is an attempt to take the scalar product of a vector and a scalar, and so is meaningless.

30. (a)
$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$
The magnitudes are both positive, so if we take the absolute value we get
$$|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos\theta|$$
and since $-1 \le \cos\theta \le 1$ it follows that
$$\begin{aligned}0 \le |\cos\theta| &\le 1\\ \therefore\quad 0 \le |\mathbf{a}||\mathbf{b}||\cos\theta| &\le |\mathbf{a}||\mathbf{b}|\\ \therefore\quad |\mathbf{a}\cdot\mathbf{b}| &\le |\mathbf{a}||\mathbf{b}|\end{aligned}$$
□

(b) Refer to question 21(b) for the first step.
$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= a^2 + 2\mathbf{a}\cdot\mathbf{b} + b^2\\ |\mathbf{a}+\mathbf{b}|^2 &= a^2 + 2\mathbf{a}\cdot\mathbf{b} + b^2\\ (|\mathbf{a}|+|\mathbf{b}|)^2 &= a^2 + 2|\mathbf{a}||\mathbf{b}| + b^2\end{aligned}$$
But
$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &\le |\mathbf{a}||\mathbf{b}|\\ \therefore\quad 2\mathbf{a}\cdot\mathbf{b} &\le 2|\mathbf{a}||\mathbf{b}|\\ \therefore\quad a^2 + 2\mathbf{a}\cdot\mathbf{b} + b^2 &\le 2a^2 + |\mathbf{a}||\mathbf{b}| + b^2\\ \therefore\quad |\mathbf{a}+\mathbf{b}|^2 &\le (|\mathbf{a}|+|\mathbf{b}|)^2\end{aligned}$$
and since all the magnitudes are positive
$$\therefore\quad |\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$$
□

Exercise 8C

1. (a) $\mathbf{a}\cdot\mathbf{b} = 3\times 5 + (-2)\times 6 = 15 - 12 = 3$
   (b) $\mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b} = 3$
   (c) $\mathbf{a}\cdot\mathbf{c} = 3\times 2 + (-2)\times(-1) = 6 + 2 = 8$
   (d) $\mathbf{b}\cdot\mathbf{c} = 5\times 2 + 6\times(-1) = 10 - 6 = 4$

2. (a) $\mathbf{x}\cdot\mathbf{y} = 2\times 5 + 3\times(-1) = 10 - 3 = 7$
   (b) $\mathbf{x}\cdot\mathbf{z} = 2\times 4 + 3\times 2 = 8 + 6 = 14$
   (c) $\mathbf{z}\cdot\mathbf{x} = \mathbf{x}\cdot\mathbf{z} = 14$
   (d) $\mathbf{y}\cdot\mathbf{z} = 5\times 4 + (-1)\times 2 = 20 - 2 = 18$

3. (a) $\mathbf{q}\cdot\mathbf{r} = 2\times 5 + (-1)\times 2 = 10 - 2 = 8$
   (b) $2\mathbf{q}\cdot 3\mathbf{r} = 2\times 2\times 3\times 5 + 2\times -1\times 3\times 2 = 60 - 12 = 48$
   Alternatively $2\mathbf{q}\cdot 3\mathbf{r} = 6\mathbf{q}\cdot\mathbf{r} = 6\times 8 = 48$
   (c) $\mathbf{p}\cdot(\mathbf{q}+\mathbf{r}) = \langle 3, 1\rangle\cdot\langle 2+5, -1+2\rangle = 3\times 7 + 1\times 1 = 22$
   (d) $\mathbf{p}\cdot(\mathbf{q}-\mathbf{r}) = \langle 3, 1\rangle\cdot\langle 2-5, -1-2\rangle = 3\times -3 + 1\times -3 = -12$

4. (a) $2\times 4 + 3\times -2 \ne 0$
   Vectors are not perpendicular.
   (b) $-2\times 4 + 1\times -2 \ne 0$
   Vectors are not perpendicular.
   (c) $3\times 2 + -1\times 6 = 0$
   Vectors are perpendicular.
   (d) $12\times 1 + -3\times 4 = 0$
   Vectors are perpendicular.
   (e) $5\times -3 + 2\times 7 \ne 0$
   Vectors are not perpendicular.

(f) $14 \times -4 + 8 \times 7 = 0$
Vectors are perpendicular.

5. (a) $3 \times 2 + 1 \times 4 = 10$
(b) $2 \times -2 + 4 \times -3 = -16$
(c) $3(2-2) + 1(4-3) = 1$
(d) $(3+2)(-2) + (1+4)(-3) = -25$

6. (a) $2(3+4) + (-1)(2-3) = 15$
(b) $(2+3)(4) + (-1+2)(-3) = 17$
(c) $3(2+4) + 2(-1-3) = 10$
(d) $$\begin{aligned}(2-3)(3-4)+&(-1-2)(2--3)\\ &= 1-15\\ &= -14\end{aligned}$$

7. (a) $\mathbf{a}\cdot\mathbf{b} = 2 \times 1 + 3 \times -4 = -10$
(b) $\mathbf{a}\cdot\mathbf{c} = 2 \times -4 + 3 \times 5 = 7$
(c) $\mathbf{b}+\mathbf{c} = (1-4)\mathbf{i} + (-4+5)\mathbf{j} = -3\mathbf{i}+\mathbf{j}$
(d) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = 2 \times -3 + 3 \times 1 = -3$
(e) $\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} = -10 + 7 = -3$

8. (a) $|\mathbf{p}| = \sqrt{3^2+4^2} = 5$
(b) $|\mathbf{q}| = \sqrt{5^2+(-12)^2} = 13$
(c) $\mathbf{p}\cdot\mathbf{q} = 3 \times 5 + 4 \times -12 = -33$
$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= |\mathbf{p}||\mathbf{q}|\cos\theta\\ -33 &= 5 \times 13\cos\theta\\ \cos\theta &= \frac{-33}{65}\\ \theta &= \cos^{-1}\frac{-33}{65}\\ &= 121^\circ\end{aligned}$$

9. (a) $|\mathbf{c}| = \sqrt{7^2+7^2} = 7\sqrt{2}$
(b) $|\mathbf{d}| = \sqrt{15^2+(-8)^2} = 17$
(c) $\mathbf{c}\cdot\mathbf{d} = 7 \times 15 + 7 \times -8 = 49$
$$\begin{aligned}\mathbf{c}\cdot\mathbf{d} &= |\mathbf{c}||\mathbf{d}|\cos\theta\\ 49 &= 7\sqrt{2} \times 17\cos\theta\\ \cos\theta &= \frac{49}{119\sqrt{2}}\\ \theta &= \cos^{-1}\frac{7}{17 sqrt2}\\ &= 73^\circ\end{aligned}$$

10. $$\begin{aligned}\mathbf{b}+\mathbf{c} &= (2+4)\mathbf{i} + (5-1)\mathbf{j}\\ &= 6\mathbf{i}+4\mathbf{j}\\ \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= 2 \times 6 + (-3) \times 4\\ &= 0\end{aligned}$$
$\therefore$ $\mathbf{a}$ is perpendicular to $(\mathbf{b}+\mathbf{c})$. □

11. $$\begin{aligned}\mathbf{a}+2\mathbf{b} &= (-2+2\times 5)\mathbf{i} + (2+2\times 2)\mathbf{j}\\ &= 8\mathbf{i}+6\mathbf{j}\\ \mathbf{b}-2\mathbf{c} &= (5-2\times 4)\mathbf{i} + (2-2\times -1)\mathbf{j}\\ &= -3\mathbf{i}+4\mathbf{j}\\ (\mathbf{a}+2\mathbf{b})\cdot(\mathbf{b}-2\mathbf{c}) &= 8\times -3 + 6 \times 4\\ &= 0\end{aligned}$$
$\therefore$ $(\mathbf{a}+2\mathbf{b})$ is perpendicular to $(\mathbf{b}-2\mathbf{c})$. □

The approach used for finding the angle in questions 12 to 17 is identical.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta\\ \cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\\ \theta &= \cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\end{aligned}$$

There seems little point in working through each of these in detail here. Question 12 is done as a sample:

12. $$\begin{aligned}|\mathbf{a}| &= \sqrt{3^2+4^2}\\ &= 5\\ |\mathbf{b}| &= \sqrt{4^2+3^2}\\ &= 5\\ \mathbf{a}\cdot\mathbf{b} &= 3\times 4 + 4 \times 3\\ &= 24\\ \theta &= \cos^{-1}\frac{24}{5\times 5}\\ &= 16^\circ\end{aligned}$$

18. For $\mathbf{a}$ and $\mathbf{b}$ to be parallel
$$\begin{aligned}\frac{3}{2} &= \frac{1}{2}\lambda\\ \lambda &= 8\end{aligned}$$
For $\mathbf{a}$ and $\mathbf{c}$ to be perpendicular
$$\begin{aligned}2\mu + 3 \times -5 &= 0\\ 2\mu &= 15\\ \mu &= 7.5\end{aligned}$$

19. For $\mathbf{d}$ and $\mathbf{e}$ to be perpendicular
$$\begin{aligned}w \times -1 + 1 \times 7 &= 0\\ w &= 7\end{aligned}$$
For $|\mathbf{d}| = |\mathbf{f}|$ with $x < 0$:
$$\begin{aligned}\sqrt{7^2+1^2} &= \sqrt{x^2+5^2}\\ 50 &= x^2+25\\ x^2 &= 25\\ x &= -5\end{aligned}$$

20. From our understanding of the vector equation of a line, we can say that L_1 and L_2 will be perpendicular if $\begin{pmatrix}-1\\3\end{pmatrix}$ and $\begin{pmatrix}6\\2\end{pmatrix}$ are perpendicular (since these vectors represent the direction of each line respectively).
$$\begin{aligned}\begin{pmatrix}-1\\3\end{pmatrix}\cdot\begin{pmatrix}6\\2\end{pmatrix} &= -1\times 6 + 3 \times 2\\ &= 0\end{aligned}$$
$\therefore$ the lines are perpendicular. □

21. $$\begin{aligned}(a\mathbf{i}+b\mathbf{j})\cdot(3\mathbf{i}-4\mathbf{j}) &= 0\\ 3a-4b &= 0\\ b &= \frac{3}{4}a\\ a^2+b^2 &= 25^2\\ a^2+\left(\frac{3}{4}a\right)^2 &= 625\\ a^2+\frac{9}{16}a^2 &= 625\\ \frac{25}{16}a^2 &= 625\\ \frac{a^2}{16} &= 25\\ a^2 &= 400\\ a &= \pm 20\\ b &= \pm 15\end{aligned}$$

The two vectors are $(20\mathbf{i}+15\mathbf{j})$ and $(-20\mathbf{i}-15\mathbf{j})$.

22. First, find any vector perpendicular to $\begin{pmatrix}3\\2\end{pmatrix}$.

$$\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}3\\2\end{pmatrix} = 0$$
$$3a+2b = 0$$

There are infinitely many solutions to this, but for the vector equation of a line we really only concern ourselves with the direction that this represents so we may choose any solution that suits us, say $a = 2$ and $b = -3$. This gives us the line

$$\mathbf{r} = \begin{pmatrix}4\\3\end{pmatrix} + \mu\begin{pmatrix}2\\-3\end{pmatrix}$$

There are infinitely many equally correct answers, although few of them as simple and obvious as this. Note that Sadler's own answer is different.

23. $$\begin{aligned}(a\mathbf{i}+b\mathbf{j})\cdot(2\mathbf{i}+\mathbf{j}) &= 0\\ 2a+b &= 0\\ b &= -2a\\ a^2+b^2 &= 1^2\\ a^2+(-2a)^2 &= 1\\ a^2+4a^2 &= 1\\ 5a^2 &= 1\\ a^2 &= \frac{1}{5}\\ a &= \pm\frac{1}{\sqrt{5}}\\ b &= \mp\frac{2}{\sqrt{5}}\end{aligned}$$

The two vectors are $\left(\frac{1}{\sqrt{5}}\mathbf{i}-\frac{2}{\sqrt{5}}\mathbf{j}\right)$ and $\left(-\frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{j}\right)$.

(We could have done this more simply: find a unit vector parallel to $\mathbf{i}-2\mathbf{j}$ or to $-\mathbf{i}+2\mathbf{j}$ . . . almost a one-step solution.)

24. Use the approach outlined for questions 12 to 17 above:

$$\begin{aligned}\left|\begin{pmatrix}1\\-4\end{pmatrix}\right| &= \sqrt{1^2+(-4)^2}\\ &= \sqrt{17}\\ \left|\begin{pmatrix}2\\1\end{pmatrix}\right| &= \sqrt{2^2+1^2}\\ &= \sqrt{5}\\ \begin{pmatrix}1\\-4\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix} &= 1\times 2+-4\times 1\\ &= -2\\ \theta &= \cos^{-1}\frac{-2}{\sqrt{17}\sqrt{5}}\\ &= 103^\circ\end{aligned}$$

but this is obtuse so we need the supplementary angle: 77°.

25. $$\begin{aligned}\overrightarrow{AC} &= (7\mathbf{i}+2\mathbf{j})-(2\mathbf{i}+4\mathbf{j})\\ &= 5\mathbf{i}-2\mathbf{j}\\ \overrightarrow{BD} &= (4\mathbf{i}+1\mathbf{j})-(6\mathbf{i}+6\mathbf{j})\\ &= -2\mathbf{i}-5\mathbf{j}\\ \overrightarrow{AC}\cdot\overrightarrow{BD} &= 5\times -2+-2\times -5\\ &= 0\end{aligned}$$

$\therefore$ $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$.

26. (a) $$\begin{aligned}\overrightarrow{AC} &= (8\mathbf{i}+9\mathbf{j})-(4\mathbf{i}+7\mathbf{j})\\ &= 4\mathbf{i}+2\mathbf{j}\end{aligned}$$

(b) $$\begin{aligned}\overrightarrow{AB} &= (6\mathbf{i}+2\mathbf{j})-(4\mathbf{i}+7\mathbf{j})\\ &= 2\mathbf{i}-5\mathbf{j}\end{aligned}$$

(c) $$\begin{aligned}\overrightarrow{AC}\cdot\overrightarrow{AB} &= 4\times 2+2\times -5\\ &= -2\end{aligned}$$

(d) $$\begin{aligned}\angle CAB &= \cos^{-1}\frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{|\overrightarrow{AC}||\overrightarrow{AB}|}\\ &= \cos^{-1}\frac{-2}{\sqrt{20}\sqrt{29}}\\ &= 95^\circ\end{aligned}$$

27. Start with the definition of dot product:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos 45^\circ\\ \cos 45^\circ &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\\ \frac{1}{\sqrt{2}} &= \frac{3\times 4+-1\times y}{\sqrt{10}\sqrt{16+y^2}}\\ \frac{1}{\sqrt{2}} &= \frac{12-y}{\sqrt{10}\sqrt{16+y^2}}\\ 1 &= \frac{12-y}{\sqrt{5}\sqrt{16+y^2}}\\ 12-y &= \sqrt{5}\sqrt{16+y^2}\end{aligned}$$

square both sides, but note that our solution must satisfy $12 - y \geq 0$.

$$\begin{aligned}(12-y)^2 &= 5(16+y^2)\\ 144 - 24y + y^2 &= 80 + 5y^2\\ 4y^2 + 24y - 64 &= 0\\ y^2 + 6y - 16 &= 0\\ (y+8)(y-2) &= 0\\ y &= -8\\ \text{or } y &= 2\end{aligned}$$

Exercise 8D

1. $$\begin{aligned}\mathbf{r}\cdot(3\mathbf{i}+4\mathbf{j}) &= (2\mathbf{i}+3\mathbf{j})\cdot(3\mathbf{i}+4\mathbf{j})\\ \mathbf{r}\cdot(3\mathbf{i}+4\mathbf{j}) &= 6+12\\ \mathbf{r}\cdot(3\mathbf{i}+4\mathbf{j}) &= 18\end{aligned}$$

2. $$\begin{aligned}\mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}) &= (-\mathbf{i}+7\mathbf{j})\cdot(5\mathbf{i}-\mathbf{j})\\ \mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}) &= -5-7\\ \mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}) &= -12\end{aligned}$$

3. $$\begin{aligned}\mathbf{r}\cdot(2\mathbf{i}+3\mathbf{j}) &= (-\mathbf{i}+2\mathbf{j})\cdot(2\mathbf{i}+3\mathbf{j})\\ \mathbf{r}\cdot(2\mathbf{i}+3\mathbf{j}) &= -2+6\\ \mathbf{r}\cdot(2\mathbf{i}+3\mathbf{j}) &= 4\end{aligned}$$

4. (A) $$\begin{aligned}(6\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) &= 0+12\\ &= 12\end{aligned}$$
   Point A lies on the line.

   (B) $$\begin{aligned}(6\mathbf{i}+3\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) &= 6+6\\ &= 12\end{aligned}$$
   Point B lies on the line.

   (C) $$\begin{aligned}(10\mathbf{i})\cdot(\mathbf{i}+2\mathbf{j}) &= 10+0\\ &= 10\end{aligned}$$
   Point C does not lie on the line.

   (D) $$\begin{aligned}(3\mathbf{i}+6\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) &= 3+12\\ &= 15\end{aligned}$$
   Point D does not lie on the line.

   (E) $$\begin{aligned}(-4\mathbf{i}+8\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) &= -4+16\\ &= 12\end{aligned}$$
   Point E lies on the line.

   (F) $$\begin{aligned}(14\mathbf{i}-\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) &= 14-2\\ &= 12\end{aligned}$$
   Point F lies on the line.

5. (G) $$\begin{aligned}(2\mathbf{i}+8\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) &= 8-24\\ &= -16\end{aligned}$$
   Point G lies on the line.

   (H) $$\begin{aligned}(4\mathbf{i}-\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) &= 16+3\\ &= 19\end{aligned}$$
   Point H does not lie on the line.

   (I) $$\begin{aligned}(8\mathbf{i}+16\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) &= 32-48\\ &= -16\end{aligned}$$
   Point I lies on the line.

   (J) $$\begin{aligned}(-\mathbf{i}+4\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) &= -4-12\\ &= -16\end{aligned}$$
   Point J lies on the line.

   (K) $$\begin{aligned}(-7\mathbf{i}-4\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) &= -28+12\\ &= -16\end{aligned}$$
   Point K lies on the line.

   (L) $$\begin{aligned}(4\mathbf{i}+8\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) &= 16-24\\ &= -8\end{aligned}$$
   Point L does not lie on the line.

6. (U) $$\begin{aligned}\begin{pmatrix}u\\2\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} &= 10\\ 2u+6 &= 10\\ u &= 2\end{aligned}$$

   (V) $$\begin{aligned}\begin{pmatrix}-10\\v\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} &= 10\\ -20+3v &= 10\\ v &= 10\end{aligned}$$

   (W) $$\begin{aligned}\begin{pmatrix}w\\-4\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} &= 10\\ 2w-12 &= 10\\ w &= 11\end{aligned}$$

   (X) $$\begin{aligned}\begin{pmatrix}x\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} &= 10\\ 2x-6 &= 10\\ x &= 8\end{aligned}$$

$$\text{(Y)}\quad \begin{pmatrix}5\\y\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix}=10$$
$$10+3y=10$$
$$y=0$$

$$\text{(Z)}\quad \begin{pmatrix}z\\6\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix}=10$$
$$2z+18=10$$
$$z=-4$$

7. (a)
$$\begin{aligned}\mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j})&=(\mathbf{i}+\mathbf{j})\cdot(5\mathbf{i}+2\mathbf{j})\\ \mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j})&=5+2\\ \mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j})&=7\end{aligned}$$
(b)
$$\begin{aligned}\mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j})&=7\\ (x\mathbf{i}+y\mathbf{j})\cdot(5\mathbf{i}+2\mathbf{j})&=7\\ 5x+2y&=7\end{aligned}$$

8. (a)
$$\begin{aligned}\mathbf{r}\cdot(2\mathbf{i}+5\mathbf{j})&=(2\mathbf{i}-\mathbf{j})\cdot(2\mathbf{i}+5\mathbf{j})\\ \mathbf{r}\cdot(2\mathbf{i}+5\mathbf{j})&=4-5\\ \mathbf{r}\cdot(2\mathbf{i}+5\mathbf{j})&=-1\end{aligned}$$
(b)
$$\begin{aligned}\mathbf{r}\cdot(2\mathbf{i}+5\mathbf{j})&=-1\\ (x\mathbf{i}+y\mathbf{j})\cdot(2\mathbf{i}+5\mathbf{j})&=-1\\ 2x+5y&=-1\end{aligned}$$

9. (a)
$$\begin{aligned}\mathbf{r}\cdot(\mathbf{i}-3\mathbf{j})&=(5\mathbf{i}+2\mathbf{j})\cdot(\mathbf{i}-3\mathbf{j})\\ \mathbf{r}\cdot(\mathbf{i}-3\mathbf{j})&=5-6\\ \mathbf{r}\cdot(\mathbf{i}-3\mathbf{j})&=-1\end{aligned}$$
(b)
$$\begin{aligned}\mathbf{r}\cdot(\mathbf{i}-3\mathbf{j})&=-1\\ (x\mathbf{i}+y\mathbf{j})\cdot(\mathbf{i}-3\mathbf{j})&=-1\\ x-3y&=-1\end{aligned}$$

10.
$$\begin{aligned}\mathbf{r}\cdot(\mathbf{i}-5\mathbf{j})&=(-2\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}-5\mathbf{j})\\ \mathbf{r}\cdot(\mathbf{i}-5\mathbf{j})&=-2-5\\ \mathbf{r}\cdot(\mathbf{i}-5\mathbf{j})&=-7\\ (x\mathbf{i}+y\mathbf{j})\cdot(\mathbf{i}-5\mathbf{j})&=-7\\ x-5y&=-7\end{aligned}$$

11.
$$\begin{aligned}\mathbf{r}\cdot(2\mathbf{i}+\mathbf{j})&=(\mathbf{i}+2\mathbf{j})\cdot(2\mathbf{i}+\mathbf{j})\\ \mathbf{r}\cdot(2\mathbf{i}+\mathbf{j})&=2+2\\ \mathbf{r}\cdot(2\mathbf{i}+\mathbf{j})&=4\\ (x\mathbf{i}+y\mathbf{j})\cdot(2\mathbf{i}+\mathbf{j})&=4\\ 2x+y&=4\end{aligned}$$

12.
$$\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\lambda(\mathbf{i}-4\mathbf{j})$$
is parallel to $(\mathbf{i}-4\mathbf{j})$ and
$$\mathbf{r}\cdot(8\mathbf{i}+2\mathbf{j})=5$$
is perpendicular to $(8\mathbf{i}+2\mathbf{j})$.
$(\mathbf{i}-4\mathbf{j})\cdot(8\mathbf{i}+2\mathbf{j})=8-8=0$ so $(\mathbf{i}-4\mathbf{j})$ is perpendicular to $(8\mathbf{i}+2\mathbf{j})$, hence
$$\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\lambda(\mathbf{i}-4\mathbf{j})$$
is perpendicular to $(8\mathbf{i}+2\mathbf{j})$ and
$$\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\lambda(\mathbf{i}-4\mathbf{j})$$
is parallel to
$$\mathbf{r}\cdot(8\mathbf{i}+2\mathbf{j})=5$$
□

13.
$$\begin{aligned}\mathbf{r}\cdot(8\mathbf{i}+5\mathbf{j})&=(-\mathbf{i}+3\mathbf{j})\cdot(8\mathbf{i}+5\mathbf{j})\\ \mathbf{r}\cdot(8\mathbf{i}+5\mathbf{j})&=-8+15\\ \mathbf{r}\cdot(8\mathbf{i}+5\mathbf{j})&=7\\ (x\mathbf{i}+y\mathbf{j})\cdot(8\mathbf{i}+5\mathbf{j})&=7\\ 8x+5y&=7\end{aligned}$$

14. L_2 is perpendicular to $(6\mathbf{i}-4\mathbf{j})$
$\therefore L_2$ is perpendicular to $0.5(6\mathbf{i}-4\mathbf{j})=(3\mathbf{i}-2\mathbf{j})$
$\therefore L_2$ is perpendicular to $\lambda(3\mathbf{i}-2\mathbf{j})$
$\therefore L_2$ is perpendicular to $\mathbf{r}=5\mathbf{i}+2\mathbf{j}+\lambda(3\mathbf{i}-2\mathbf{j})$
$\therefore L_2$ is perpendicular to L_1. □

15.
$$\begin{aligned}3x+4y&=7\\ (x\mathbf{i}+y\mathbf{j})\cdot(3\mathbf{i}+4\mathbf{j})&=7\end{aligned}$$
The line is perpendicular to $(3\mathbf{i}+4\mathbf{j})$. This vector has a magnitude of 5, so a unit vector with this direction is $(0.6\mathbf{i}+0.8\mathbf{j})$.

16.
$$\begin{aligned}-5x+12y&=11\\ (x\mathbf{i}+y\mathbf{j})\cdot(-5\mathbf{i}+12\mathbf{j})&=11\end{aligned}$$
The line is perpendicular to $(-5\mathbf{i}+12\mathbf{j})$. This vector has a magnitude of 13, so the unit vectors with this direction are $\pm\frac{1}{13}(-5\mathbf{i}+12\mathbf{j})$.

17.
$$\begin{aligned}3x-y&=5\\ (x\mathbf{i}+y\mathbf{j})\cdot(3\mathbf{i}-\mathbf{j})&=5\end{aligned}$$
The line is perpendicular to $(3\mathbf{i}-1\mathbf{j})$. This vector has a magnitude of $\sqrt{10}$, so the unit vectors with this direction are $\pm\frac{1}{\sqrt{10}}(3\mathbf{i}-1\mathbf{j})$.

18. L_1 is parallel to $\begin{pmatrix}1\\-2\end{pmatrix}$ so it is also parallel to $-\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}$.
L_2 is perpendicular to $\begin{pmatrix}-1\\2\end{pmatrix}$
$\therefore L_1$ is perpendicular to L_2. □

$$\begin{aligned}\left(\begin{pmatrix}2\\8\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\2\end{pmatrix}&=4\\ \begin{pmatrix}2+\lambda\\8-2\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\2\end{pmatrix}&=4\\ -1(2+\lambda)+2(8-2\lambda)&=4\\ -2-\lambda+16-4\lambda&=4\\ 14-5\lambda&=4\\ \lambda&=2\end{aligned}$$

$$\begin{aligned}\mathbf{r}&=\begin{pmatrix}2\\8\end{pmatrix}+2\begin{pmatrix}1\\-2\end{pmatrix}\\ &=\begin{pmatrix}2\\8\end{pmatrix}+\begin{pmatrix}2\\-4\end{pmatrix}\\ &=\begin{pmatrix}4\\4\end{pmatrix}\end{aligned}$$

Exercise 8E

1. Let A be the point of closest approach.
   Let O be the position at 8am as illustrated.
   Let t be the time of closest approach in hours after 8am.

$$\begin{aligned}\overrightarrow{PA} &= \overrightarrow{PO} + \overrightarrow{OA}\\ &= \overrightarrow{OA} - \overrightarrow{OP}\\ &= t(10\mathbf{i} + 5\mathbf{j}) - (25\mathbf{j} + 15\mathbf{j})\end{aligned}$$

   Now because $\overrightarrow{PA}$ is perpendicular to $(10\mathbf{i} + 5\mathbf{j})$,

$$\begin{aligned}\overrightarrow{PA} \cdot (10\mathbf{i} + 5\mathbf{j}) &= 0\\ [t(10\mathbf{i} + 5\mathbf{j}) - (25\mathbf{j} + 15\mathbf{j})] \cdot (10\mathbf{i} + 5\mathbf{j}) &= 0\\ [t(10\mathbf{i} + 5\mathbf{j})] \cdot (10\mathbf{i} + 5\mathbf{j}) - (25\mathbf{j} + 15\mathbf{j}) \cdot (10\mathbf{i} + 5\mathbf{j}) &= 0\\ 125t - 325 &= 0\\ t &= \frac{13}{5}\end{aligned}$$

   That is, the closest approach occurs 2.6 hours after 8am, or 10:36am. At that time

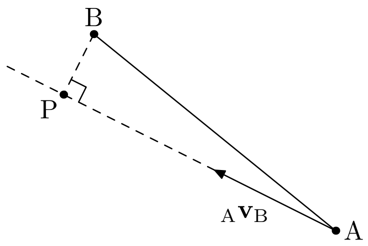
$$\begin{aligned}\overrightarrow{PA} &= \frac{13}{5}(10\mathbf{i} + 5\mathbf{j}) - (25\mathbf{j} + 15\mathbf{j})\\ &= \mathbf{i} - 2\mathbf{j}\\ PA &= \sqrt{1^2 + 2^2}\\ &= \sqrt{5}\end{aligned}$$

   The closest approach is $\sqrt{5} \approx 2.24$km.

2. Apply a velocity of $-\mathbf{v}_B$ so as to treat B as stationary.

$$\begin{aligned}{}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B\\ &= \begin{pmatrix}-10\\-2\end{pmatrix} - \begin{pmatrix}-2\\-6\end{pmatrix}\\ &= \begin{pmatrix}-8\\4\end{pmatrix}\end{aligned}$$

   Let P be the point of closest approach of particle A towards B.



$$\begin{aligned}\overrightarrow{PB} &= \overrightarrow{PA} + \overrightarrow{AB}\\ &= \overrightarrow{AB} - \overrightarrow{AP}\\ &= \begin{pmatrix}-16\\13\end{pmatrix} - t\,{}_A\mathbf{v}_B\\ &= \begin{pmatrix}-16\\13\end{pmatrix} - t\begin{pmatrix}-8\\4\end{pmatrix}\end{aligned}$$

   Now because $\overrightarrow{PB}$ is perpendicular to ${}_A\mathbf{v}_B$,

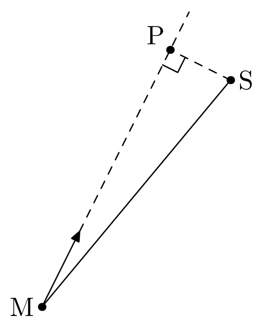
$$\begin{aligned}\overrightarrow{PB} \cdot {}_A\mathbf{v}_B &= 0\\ \left[\begin{pmatrix}-16\\13\end{pmatrix} - t\begin{pmatrix}-8\\4\end{pmatrix}\right] \cdot \begin{pmatrix}-8\\4\end{pmatrix} &= 0\\ \begin{pmatrix}-16\\13\end{pmatrix} \cdot \begin{pmatrix}-8\\4\end{pmatrix} - t\begin{pmatrix}-8\\4\end{pmatrix} \cdot \begin{pmatrix}-8\\4\end{pmatrix} &= 0\\ (-16 \times -8 + 13 \times 4) - t(-8 \times -8 + 4 \times 4) &= 0\\ 180 + 80t &= 0\\ t &= \frac{9}{4}\end{aligned}$$

   The closest approach occurs at $t = 2.25$ seconds. At that time

$$\begin{aligned}\overrightarrow{PB} &= \begin{pmatrix}-16\\13\end{pmatrix} - \frac{9}{4}\begin{pmatrix}-8\\4\end{pmatrix}\\ &= \begin{pmatrix}2\\4\end{pmatrix}\\ PB &= \sqrt{2^2 + 4^2}\\ &= 2\sqrt{5}\end{aligned}$$

   The closest approach is $2\sqrt{5} \approx 4.47$m.

3. Let M be the initial position of the mouse.
   Let S be the initial position of the snake.
   Let P be the position of closest approach.



$$\begin{aligned}\overrightarrow{PS} &= \overrightarrow{PM} + \overrightarrow{MS}\\ &= -t(\mathbf{i} + 2\mathbf{j}) + (5\mathbf{i} + 6\mathbf{j})\end{aligned}$$

$$\begin{aligned}\overrightarrow{PS} \cdot (\mathbf{i} + 2\mathbf{j}) &= 0\\ [-t(\mathbf{i} + 2\mathbf{j}) + (5\mathbf{i} + 6\mathbf{j})] \cdot (\mathbf{i} + 2\mathbf{j}) &= 0\\ -t(\mathbf{i} + 2\mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j}) + (5\mathbf{i} + 6\mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j}) &= 0\\ -5t + 17 &= 0\\ t &= \frac{17}{5}\end{aligned}$$

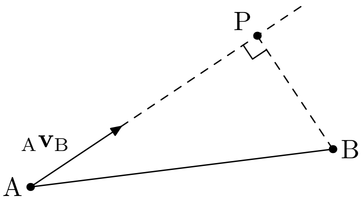
$$\begin{aligned}\overrightarrow{PS} &= -\frac{17}{5}(\mathbf{i} + 2\mathbf{j}) + (5\mathbf{i} + 6\mathbf{j})\\ &= (1.6\mathbf{i} - 0.8\mathbf{j})\\ PS &= \sqrt{1.6^2 + (-0.8)^2}\\ &= 1.8\text{m}\end{aligned}$$

The snake is more likely to catch the mouse than miss it.

4. Apply a velocity of $-\mathbf{v}_{\text{B}}$ so as to treat B as stationary.

$$\begin{aligned}{}_{\text{A}}\mathbf{v}_{\text{B}} &= \mathbf{v}_{\text{A}} - \mathbf{v}_{\text{B}}\\ &= (3\mathbf{i}+4\mathbf{j})-(-3\mathbf{i})\\ &= (6\mathbf{i}+4\mathbf{j})\text{cm/s}\end{aligned}$$

Let P be the point of closest approach of particle A towards B.



$$\begin{aligned}\overrightarrow{\text{PB}} &= \overrightarrow{\text{PA}}+\overrightarrow{\text{AB}}\\ &= -t\,{}_{\text{A}}\mathbf{v}_{\text{B}}+\overrightarrow{\text{AB}}\\ &= -t(6\mathbf{i}+4\mathbf{j})+(40\mathbf{i}+5\mathbf{j})\end{aligned}$$

Now because $\overrightarrow{\text{PB}}$ is perpendicular to ${}_{\text{A}}\mathbf{v}_{\text{B}}$,

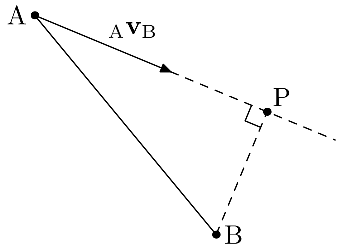
$$\begin{aligned}\overrightarrow{\text{PB}}\cdot{}_{\text{A}}\mathbf{v}_{\text{B}} &= 0\\ [-t(6\mathbf{i}+4\mathbf{j})+(40\mathbf{i}+5\mathbf{j})]\cdot(6\mathbf{i}+4\mathbf{j}) &= 0\\ -t(6\mathbf{i}+4\mathbf{j})\cdot(6\mathbf{i}+4\mathbf{j})+(40\mathbf{i}+5\mathbf{j})\cdot(6\mathbf{i}+4\mathbf{j}) &= 0\\ -52t+260 &= 0\\ t &= 5\end{aligned}$$

The closest approach occurs at $t=5$ seconds. At that time

$$\begin{aligned}\overrightarrow{\text{PB}} &= -5(6\mathbf{i}+4\mathbf{j})+(40\mathbf{i}+5\mathbf{j})\\ &= 10\mathbf{i}-15\mathbf{j}\\ \text{PB} &= 5\sqrt{13}\end{aligned}$$

The closest approach is $5\sqrt{13}\approx 18.0$cm.

5.
$$\begin{aligned}\overrightarrow{\text{AB}} &= \mathbf{r}_{\text{B}}-\mathbf{r}_{\text{A}}\\ &= \begin{pmatrix}54\\-19\end{pmatrix}-\begin{pmatrix}30\\10\end{pmatrix}\\ &= \begin{pmatrix}24\\-29\end{pmatrix}\\ {}_{\text{A}}\mathbf{v}_{\text{B}} &= \mathbf{v}_{\text{A}}-\mathbf{v}_{\text{B}}\\ &= \begin{pmatrix}10\\-5\end{pmatrix}-\begin{pmatrix}-8\\7\end{pmatrix}\\ &= \begin{pmatrix}18\\-12\end{pmatrix}\end{aligned}$$



$$\begin{aligned}\overrightarrow{\text{PB}} &= \overrightarrow{\text{PA}}+\overrightarrow{\text{AB}}\\ &= -t_{\text{A}}\mathbf{v}_{\text{B}}+\overrightarrow{\text{AB}}\\ &= -t\begin{pmatrix}18\\-12\end{pmatrix}+\begin{pmatrix}24\\-29\end{pmatrix}\end{aligned}$$

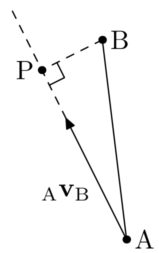
Now because $\overrightarrow{\text{PB}}$ is perpendicular to ${}_{\text{A}}\mathbf{v}_{\text{B}}$,

$$\begin{aligned}\overrightarrow{\text{PB}}\cdot{}_{\text{A}}\mathbf{v}_{\text{B}} &= 0\\ \left[-t\begin{pmatrix}18\\-12\end{pmatrix}+\begin{pmatrix}24\\-29\end{pmatrix}\right]\cdot\begin{pmatrix}18\\-12\end{pmatrix} &= 0\\ -t\begin{pmatrix}18\\-12\end{pmatrix}\cdot\begin{pmatrix}18\\-12\end{pmatrix}+\begin{pmatrix}24\\-29\end{pmatrix}\cdot\begin{pmatrix}18\\-12\end{pmatrix} &= 0\\ -468t+780 &= 0\\ t &= \frac{5}{3}\end{aligned}$$

The closest approach occurs at $t=\frac{5}{3}$ hours (i.e. at 4:40am). At that time

$$\begin{aligned}\overrightarrow{\text{PB}} &= -\frac{5}{3}\begin{pmatrix}18\\-12\end{pmatrix}+\begin{pmatrix}24\\-29\end{pmatrix}\\ &= \begin{pmatrix}6\\-9\end{pmatrix}\\ \text{PB} &= 3\sqrt{13}\text{km}\end{aligned}$$

6.
$$\begin{aligned}\overrightarrow{\text{AB}} &= \mathbf{r}_{\text{B}}-\mathbf{r}_{\text{A}}\\ &= (16\mathbf{i}+23\mathbf{j})-(20\mathbf{i}-10\mathbf{j})\\ &= (-4\mathbf{i}+33\mathbf{j})\\ {}_{\text{A}}\mathbf{v}_{\text{B}} &= \mathbf{v}_{\text{A}}-\mathbf{v}_{\text{B}}\\ &= (4\mathbf{i}+5\mathbf{j})-(6\mathbf{i}-3\mathbf{j})\\ &= (-2\mathbf{i}+8\mathbf{j})\end{aligned}$$



$$\begin{aligned}\overrightarrow{\text{PB}} &= \overrightarrow{\text{PA}}+\overrightarrow{\text{AB}}\\ &= -t_{\text{A}}\mathbf{v}_{\text{B}}+\overrightarrow{\text{AB}}\\ &= -t\,(-2\mathbf{i}+8\mathbf{j})+(-4\mathbf{i}+33\mathbf{j})\end{aligned}$$

Because $\overrightarrow{\text{PB}}$ is perpendicular to ${}_{\text{A}}\mathbf{v}_{\text{B}}$,

$$\begin{aligned}\overrightarrow{\text{PB}}\cdot{}_{\text{A}}\mathbf{v}_{\text{B}} &= 0\\ [-t\,(-2\mathbf{i}+8\mathbf{j})+(-4\mathbf{i}+33\mathbf{j})]\cdot(-2\mathbf{i}+8\mathbf{j}) &= 0\\ -t\,(-2\mathbf{i}+8\mathbf{j})\cdot(-2\mathbf{i}+8\mathbf{j})+(-4\mathbf{i}+33\mathbf{j})\cdot\begin{pmatrix}-2\\8\end{pmatrix} &= 0\\ -68t+272 &= 0\\ t &= 4\end{aligned}$$

The closest approach occurs at $t = 4$ seconds. At that time

$$\begin{aligned}\overrightarrow{\text{PB}} &= -4\left(-2\mathbf{i}+8\mathbf{j}\right)+\left(-4\mathbf{i}+33\mathbf{j}\right)\\ &= \left(4\mathbf{i}+1\mathbf{j}\right)\\ \text{PB} &= \sqrt{17}\text{m}\end{aligned}$$

7. Let P be the point where the perpendicular from A meets line L.

$$\begin{aligned}\overrightarrow{\text{OP}} &= -5\mathbf{i}+22\mathbf{j}+\lambda_1\left(5\mathbf{i}-2\mathbf{j}\right)\\ \overrightarrow{\text{AP}} &= \overrightarrow{\text{AO}}+\overrightarrow{\text{OP}}\\ &= -\left(14\mathbf{i}-3\mathbf{j}\right)-5\mathbf{i}+22\mathbf{j}+\lambda_1\left(5\mathbf{i}-2\mathbf{j}\right)\\ &= (5\lambda_1-19)\mathbf{i}+(-2\lambda_1+25)\mathbf{j}\end{aligned}$$

Because $\overrightarrow{\text{AP}}$ is perpendicular to L,

$$\begin{aligned}(5\mathbf{i}-2\mathbf{j})\cdot(5\lambda_1-19)\mathbf{i}+(-2\lambda_1+25)\mathbf{j} &= 0\\ 25\lambda_1-95+4\lambda_1-50 &= 0\\ 29\lambda_1 &= 145\\ \lambda_1 &= 5\end{aligned}$$

$$\begin{aligned}\therefore\qquad \overrightarrow{\text{AP}} &= (5\times5-19)\mathbf{i}+(-2\times5+25)\mathbf{j}\\ &= 6\mathbf{i}+15\mathbf{j}\\ \text{AP} &= 3\sqrt{29}\end{aligned}$$

8. Let P be the point where the perpendicular from A meets line L.

$$\begin{aligned}\overrightarrow{\text{OP}} &= \begin{pmatrix}3\\-1\end{pmatrix}+\lambda_1\begin{pmatrix}3\\4\end{pmatrix}\\ \overrightarrow{\text{AP}} &= \overrightarrow{\text{AO}}+\overrightarrow{\text{OP}}\\ &= -\begin{pmatrix}11\\18\end{pmatrix}+\begin{pmatrix}3\\-1\end{pmatrix}+\lambda_1\begin{pmatrix}3\\4\end{pmatrix}\\ &= \begin{pmatrix}-8\\-19\end{pmatrix}+\lambda_1\begin{pmatrix}3\\4\end{pmatrix}\end{aligned}$$

Because $\overrightarrow{\text{AP}}$ is perpendicular to L,

$$\begin{aligned}\begin{pmatrix}3\\4\end{pmatrix}\cdot\left[\begin{pmatrix}-8\\-19\end{pmatrix}+\lambda_1\begin{pmatrix}3\\4\end{pmatrix}\right] &= 0\\ \begin{pmatrix}3\\4\end{pmatrix}\cdot\begin{pmatrix}-8\\-19\end{pmatrix}+\lambda_1\begin{pmatrix}3\\4\end{pmatrix}\cdot\begin{pmatrix}3\\4\end{pmatrix} &= 0\\ -100+25\lambda_1 &= 0\\ \lambda_1 &= 4\end{aligned}$$

$$\begin{aligned}\therefore\qquad \overrightarrow{\text{AP}} &= \begin{pmatrix}-8\\-19\end{pmatrix}+4\begin{pmatrix}3\\4\end{pmatrix}\\ &= \begin{pmatrix}4\\-3\end{pmatrix}\\ \text{AP} &= 5\end{aligned}$$

9. Let P be the point where the perpendicular from A meets line L.

$$\begin{aligned}\overrightarrow{\text{OP}} &= \begin{pmatrix}-3\\0\end{pmatrix}+\lambda_1\begin{pmatrix}2\\-2\end{pmatrix}\\ \overrightarrow{\text{AP}} &= \overrightarrow{\text{AO}}+\overrightarrow{\text{OP}}\\ &= -\begin{pmatrix}-3\\8\end{pmatrix}+\begin{pmatrix}-3\\0\end{pmatrix}+\lambda_1\begin{pmatrix}2\\-2\end{pmatrix}\\ &= \begin{pmatrix}0\\-8\end{pmatrix}+\lambda_1\begin{pmatrix}2\\-2\end{pmatrix}\end{aligned}$$

Because $\overrightarrow{\text{AP}}$ is perpendicular to L,

$$\begin{aligned}\begin{pmatrix}2\\-2\end{pmatrix}\cdot\left[\begin{pmatrix}0\\-8\end{pmatrix}+\lambda_1\begin{pmatrix}2\\-2\end{pmatrix}\right] &= 0\\ \begin{pmatrix}2\\-2\end{pmatrix}\cdot\begin{pmatrix}0\\-8\end{pmatrix}+\lambda_1\begin{pmatrix}2\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\-2\end{pmatrix} &= 0\\ 16+8\lambda_1 &= 0\\ \lambda_1 &= -2\end{aligned}$$

$$\begin{aligned}\therefore\qquad \overrightarrow{\text{AP}} &= \begin{pmatrix}0\\-8\end{pmatrix}-2\begin{pmatrix}2\\-2\end{pmatrix}\\ &= \begin{pmatrix}-4\\-4\end{pmatrix}\\ \text{AP} &= 4\sqrt{2}\end{aligned}$$

10. (a) Substitute $\mathbf{r} = \begin{pmatrix}a\\b\end{pmatrix}$ into the equation of the line:

$$\begin{aligned}\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}-1\\2\end{pmatrix} &= 20\\ -a+2b &= 20\\ a &= 2b-20\end{aligned}$$

(b)

$$\begin{aligned}\overrightarrow{\text{AP}} &= \overrightarrow{\text{AO}}+\overrightarrow{\text{OP}}\\ &= -\begin{pmatrix}13\\4\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}\\ &= \begin{pmatrix}a-13\\b-4\end{pmatrix}\end{aligned}$$

(c) $\begin{pmatrix}-1\\2\end{pmatrix}$ is perpendicular to the line L, so if $\overrightarrow{\text{AP}}$ is to also be perpendicular to L then it must be parallel to $\begin{pmatrix}-1\\2\end{pmatrix}$.

(d) First substitute for a:

$$\begin{aligned}\overrightarrow{\text{AP}} &= \begin{pmatrix}a-13\\b-4\end{pmatrix}\\ &= \begin{pmatrix}2b-33\\b-4\end{pmatrix}\end{aligned}$$

Now because $\overrightarrow{\text{AP}}$ is parallel to $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\begin{aligned}
\begin{pmatrix} 2b-33 \\ b-4 \end{pmatrix} &= k\begin{pmatrix} -1 \\ 2 \end{pmatrix} \\
2b-33 &= -k \\
\text{and} \qquad b-4 &= 2k \\
4b-66 &= -2k \\
5b-70 &= 0 \\
b &= 14 \\
a &= 2b = 20 \\
&= 8 \\
\overrightarrow{\text{AP}} &= \begin{pmatrix} a-13 \\ b-4 \end{pmatrix} \\
&= \begin{pmatrix} 8-13 \\ 14-4 \end{pmatrix} \\
&= \begin{pmatrix} -5 \\ 10 \end{pmatrix} \\
|\overrightarrow{\text{AP}}| &= 5\sqrt{5}
\end{aligned}$$

11. Let P$=\begin{pmatrix} p \\ q \end{pmatrix}$ be the point where the perpendicular from A$=\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ meets the line. Substitute $\mathbf{r} = \begin{pmatrix} p \\ q \end{pmatrix}$ into the equation of the line:

$$\begin{aligned}
\begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} 0 \\ c \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \\
p &= \lambda \\
q &= c + \lambda m \\
&= c + mp
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\text{AP}} &= \overrightarrow{\text{AO}} + \overrightarrow{\text{OP}} \\
&= -\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} \\
&= \begin{pmatrix} p-x_1 \\ q-y_1 \end{pmatrix}
\end{aligned}$$

$\begin{pmatrix} 1 \\ m \end{pmatrix}$ is parallel to the line, so a vector perpendicular to the line is $\begin{pmatrix} -m \\ 1 \end{pmatrix}$. If $\overrightarrow{\text{AP}}$ is to also be perpendicular to the line then it must be parallel to $\begin{pmatrix} -m \\ 1 \end{pmatrix}$. First substitute for q:

$$\begin{aligned}
\overrightarrow{\text{AP}} &= \begin{pmatrix} p-x_1 \\ q-y_1 \end{pmatrix} \\
&= \begin{pmatrix} p-x_1 \\ c+mp-y_1 \end{pmatrix}
\end{aligned}$$

Now because $\overrightarrow{\text{AP}}$ is parallel to $\begin{pmatrix} -m \\ 1 \end{pmatrix}$

$$\begin{aligned}
\begin{pmatrix} p-x_1 \\ c+mp-y_1 \end{pmatrix} &= k\begin{pmatrix} -m \\ 1 \end{pmatrix} \\
p-x_1 &= -mk \\
\text{and} \qquad c+mp-y_1 &= k \\
pm - mx_1 &= -m^2k \\
c - y_1 + mx_1 &= k + m^2k \\
mx_1 - y_1 + c &= k(m^2+1) \\
k &= \frac{mx_1-y_1+c}{m^2+1} \\
\overrightarrow{\text{AP}} &= k\begin{pmatrix} -m \\ 1 \end{pmatrix} \\
&= \frac{mx_1-y_1+c}{1+m^2}\begin{pmatrix} -m \\ 1 \end{pmatrix} \\
|\overrightarrow{\text{AP}}| &= \left|\frac{mx_1-y_1+c}{1+m^2}\right|\sqrt{m^2+1} \\
&= \left|\frac{mx_1-y_1+c}{\sqrt{m^2+1}}\right|
\end{aligned}$$

If we represent the line as $ax+by+d=0$, this translates to the vector equation

$$\mathbf{r}\cdot\begin{pmatrix} a \\ b \end{pmatrix} = -d$$

where $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$, so the line is perpendicular to $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\overrightarrow{\text{AP}}$ is parallel to $\begin{pmatrix} a \\ b \end{pmatrix}$.

As before,

$$\overrightarrow{\text{AP}} = \begin{pmatrix} p-x_1 \\ q-y_1 \end{pmatrix}$$

so

$$\begin{aligned}
\begin{pmatrix} p-x_1 \\ q-y_1 \end{pmatrix} &= k\begin{pmatrix} a \\ b \end{pmatrix} \\
p-x_1 &= ak \\
\text{and} \qquad q-y_1 &= bk \\
ap - ax_1 &= a^2k \\
bq - by_1 &= b^2k \\
ap + bq - ax_1 - by_1 &= k(a^2+b^2)
\end{aligned}$$

but since P is a point on the line, $ap+bq+d=0$ or $ap+bq=-d$

$$\begin{aligned}
-d - ax_1 - by_1 &= k(a^2+b^2) \\
k &= -\frac{ax_1+by_1+d}{a^2+b^2} \\
\overrightarrow{\text{AP}} &= k\begin{pmatrix} a \\ b \end{pmatrix} \\
&= -\frac{ax_1+by_1+d}{a^2+b^2}\begin{pmatrix} a \\ b \end{pmatrix} \\
|\overrightarrow{\text{AP}}| &= \left|-\frac{ax_1+by_1+d}{a^2+b^2}\right|\sqrt{a^2+b^2} \\
&= \left|\frac{ax_1+by_1+d}{\sqrt{a^2+b^2}}\right|
\end{aligned}$$

Miscellaneous Exercise 8

1. Cosine is positive in the 1st and 4th quadrants, so the smallest solution in the given domain is $x = 180 - 132 = 48$. The next is $x = 360 - 48 = 312$, then $x = 360 + 48 = 408$ and finally $x = 720 - 48 = 672$.

2. (a)
$$\begin{aligned} \mathbf{c}\cdot\mathbf{d} &= |\mathbf{c}||\mathbf{d}|\cos\theta \\ -5 &= 2\times 3\cos\theta \\ \cos\theta &= -\frac{5}{6} \\ \theta &= \cos^{-1} -\frac{5}{6} \\ &= 146^\circ \end{aligned}$$

   (b) $\mathbf{c}\cdot\mathbf{c} = |\mathbf{c}|^2 = 2^2 = 4$

   (c) $\mathbf{d}\cdot\mathbf{d} = |\mathbf{d}|^2 = 3^2 = 9$

   (d)
$$\begin{aligned} (\mathbf{c}+\mathbf{d})\cdot(\mathbf{c}+\mathbf{d}) &= \mathbf{c}\cdot\mathbf{c} + 2\mathbf{c}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{d} \\ &= 4 + 2\times(-5) + 9 \\ &= 3 \end{aligned}$$

   (e)
$$\begin{aligned} |\mathbf{c}+\mathbf{d}|^2 &= (\mathbf{c}+\mathbf{d})\cdot(\mathbf{c}+\mathbf{d}) \\ &= 3 \\ \therefore\quad |\mathbf{c}+\mathbf{d}| &= \sqrt{3} \end{aligned}$$

3. (a) True. $\mathbf{a}$ and $(\mathbf{b}-\mathbf{a})$ are perpendicular vectors so they must have a zero scalar product.

   (b) True. Proof:
$$\begin{aligned} \mathbf{a}\cdot(\mathbf{b}-\mathbf{a}) &= 0 && \text{(see part a)} \\ \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{a} &= 0 && \text{(expanding)} \\ \mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot\mathbf{a} \end{aligned}$$
□

   (c) Not necessarily true. $\mathbf{a}\cdot\mathbf{b} = k$ is the vector equation of a line perpendicular to $\mathbf{a}$ such that every point on the line can be substituted for $\mathbf{b}$ to satisfy the equation so we cannot conclude that $\mathbf{b} = \mathbf{a}$.

   (d) True. This follows from part (b).

4. Substitute $u = \frac{1}{2}\theta$. The L.H.S. becomes:
$$\begin{aligned} \frac{\sin 2u}{\cos u} &= \frac{2\sin u\cos u}{\cos u} \\ &= 2\sin u \\ &= 2\sin\left(\frac{1}{2}\theta\right) \\ &= \text{R.H.S.} \end{aligned}$$
□

5. (a) $\overrightarrow{\text{ED}} = \overrightarrow{\text{OD}} - \overrightarrow{\text{OE}} = -6\mathbf{i} + \mathbf{j}$

   (b) $\overrightarrow{\text{EF}} = \overrightarrow{\text{OF}} - \overrightarrow{\text{OE}} = 5\mathbf{i} + 5\mathbf{j}$

   (c) $\overrightarrow{\text{ED}}\cdot\overrightarrow{\text{EF}} = -6\times 5 + 1\times 5 = -25$

   (d)
$$\begin{aligned} \overrightarrow{\text{ED}}\cdot\overrightarrow{\text{EF}} &= |\overrightarrow{\text{ED}}||\overrightarrow{\text{EF}}|\cos\angle\text{DEF} \\ \cos\angle\text{DEF} &= \frac{\overrightarrow{\text{ED}}\cdot\overrightarrow{\text{EF}}}{|\overrightarrow{\text{ED}}||\overrightarrow{\text{EF}}|} \\ &= \frac{-25}{\sqrt{6^2+1^2}\sqrt{5^2+5^2}} \\ &= \frac{-25}{\sqrt{37}\sqrt{50}} \\ \angle\text{DEF} &= \cos^{-1}\frac{-25}{\sqrt{37}\sqrt{50}} \\ &= 126^\circ \end{aligned}$$

6. (a) Already in the right form to read centre and radius directly. Centre= $(3, -2)$; radius= 7.

   (b) $|\mathbf{r} - (2\mathbf{i} + 7\mathbf{j})| = 11$.
   Centre= $(2, 7)$; radius= 11.

   (c) $(x-3)^2 + (y - -2)^2 = 4^2$
   Centre= $(3, -2)$; radius= 4.

   (d) $(x - -1)^2 + (y - -7)^2 = (2\sqrt{5})^2$
   Centre= $(-1, -7)$, radius=$2\sqrt{5}$.

   (e)
$$\begin{aligned} x^2 + y^2 - 8x &= 4y + 5 \\ x^2 - 8x + y^2 - 4y &= 5 \\ (x-4)^2 - 16 + (y-2)^2 - 4 &= 5 \\ (x-4)^2 + (y-2)^2 &= 25 \\ &= 5^2 \end{aligned}$$
   Centre= $(4, 2)$; radius= 5.

   (f)
$$\begin{aligned} x^2 + 6x + y^2 - 14y &= 42 \\ (x+3)^2 - 9 + (y-7)^2 - 49 &= 42 \\ (x - -3)^2 + (y-7)^2 &= 100 \\ &= 10^2 \end{aligned}$$
   Centre= $(-3, 7)$; radius= 10.

7. If solutions are not real then the discriminant $\Delta = b^2 - 4ac$ must be negative:
$$\begin{aligned} p^2 - 4\times 5\times 10 &< 0 \\ p^2 - 200 &< 0 \\ p^2 &< 200 \\ -10\sqrt{2} < p &< 10\sqrt{2} \end{aligned}$$

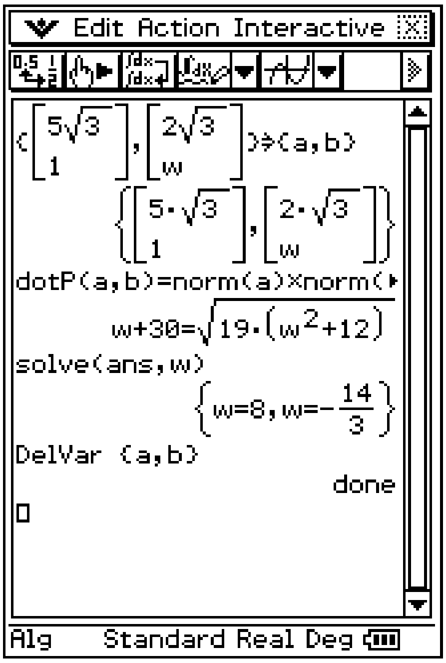
8. L_1 is parallel to $3\mathbf{i} - \mathbf{j}$ and L_2 is parallel to $2\mathbf{i} + 3\mathbf{j}$ so the angle between these vectors is the angle between the lines.
$$\begin{aligned} \cos\theta &= \frac{(3\mathbf{i}-\mathbf{j})\cdot(2\mathbf{i}+3\mathbf{j})}{|3\mathbf{i}-\mathbf{j}||2\mathbf{i}+3\mathbf{j}|} \\ &= \frac{3}{\sqrt{10}\sqrt{13}} \\ \theta &= \cos^{-1}\frac{3}{\sqrt{10}\sqrt{13}} \\ &= 75^\circ \end{aligned}$$

9. Each of the pieces is internally continuous, so we need only concern ourselves with the endpoints

of the pieces.

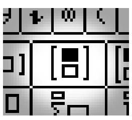
$$\begin{aligned} f(-1) &= \lim_{x\to -1^+} f(x) \\ 5(-1)+6 &= 2(-1)+a \\ a &= 3 \\ \lim_{x\to 5^-} f(x) &= f(5) \\ 2(5)+a &= b \\ b &= 13 \\ f(5) &= \lim_{x\to 5^+} f(x) \\ 13 &= 4(5)+c \\ c &= -7 \end{aligned}$$

10. This is very straightforward if you know how to use your calculator efficiently.



Going through this line by line:

- Use the curly brackets to make a list if you want to assign values to several variables at once. The curly brackets are on the `mth` tab or the `2D` tab. You could equally well have assigned the two variables individually; it makes little difference, but this way is a little more concise. (We could have done it without assigning variables at all by entering the vector values directly in the equation we want to solve, but this way is easier to follow.)
- To enter the values as column vectors, select the `CALC` page of the `2D` tab and choose the second icon:



- For the dot product of two vectors, use the `dotP` function from the `Action→Vector` menu and separate the two vectors with a comma.
- For the magnitude of a vector use the `norm` function under the `Action→Vector` menu. You can not use the absolute value `|a|` as this does not give the magnitude of a vector on the ClassPad.
- The third line in full reads `dotP(a,b)=norm(a)×norm(b)×cos(60)`
- I entered the equation I wanted to solve on one line and then solved it on the next line. This allows the whole thing to show on the calculator display which is useful for demonstration purposes, but we could equally well have done this all in one long line: `solve(dotP(a,b)=norm(a)×norm(b)×cos(60),w)`
- If your calculator is in `Cplx` mode you will get a warning that other solutions may exist. Because we are only interested in real solutions you may safely ignore this warning and it goes away if you switch to `Real` mode.
- The `DelVar` at the end deletes the variables we assigned. Although this is not strictly necessary it's not bad practice to delete variables when you've finished with them as it can reduce the occurance of problems if you later wish to re-use the variables used here without doing a `Clear All Variables` first.

You should also be able to do most (if not all) of this without a calculator:

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta \\ \mathbf{a}\cdot\mathbf{b} &= (5\sqrt{3}\mathbf{i}+\mathbf{j})\cdot(2\sqrt{3}\mathbf{i}+w\mathbf{j}) \\ &= 5\sqrt{3}\times 2\sqrt{3}+1\times w \\ &= 30+w \end{aligned}$$

$$\begin{aligned} |\mathbf{a}||\mathbf{b}|\cos\theta &= |5\sqrt{3}\mathbf{i}+\mathbf{j}||2\sqrt{3}\mathbf{i}+w\mathbf{j}|\cos 60^\circ \\ &= \sqrt{(5\sqrt{3})^2+1^2}\times\sqrt{(2\sqrt{3})^2+w^2}\times\frac{1}{2} \\ &= \sqrt{76}\sqrt{12+w^2}\frac{1}{\sqrt{4}} \\ &= \sqrt{\frac{76(12+w^2)}{4}} \\ &= \sqrt{228+19w^2} \end{aligned}$$

$$\begin{aligned} \therefore\quad 30+w &= \sqrt{228+19w^2} \\ (30+w)^2 &= 228+19w^2 \end{aligned}$$

(noting that $w + 30 \geq 0$ so we discard any solution that gives $w < -30$)

$$\begin{aligned} 900 + 60w + w^2 &= 228 + 19w^2 \\ 18w^2 - 60w - 672 &= 0 \\ 3w^2 - 10w - 112 &= 0 \end{aligned}$$

Factors of $3 \times -112 = -336$ that add to -10 are -24 and 14

$$\begin{aligned} 3w^2 - 24w + 14w - 112 &= 0 \\ 3w(w-8) + 14(w-8) &= 0 \\ (3w+14)(w-8) &= 0 \\ w &= -\frac{14}{3} \\ \text{or} \quad w &= 8 \end{aligned}$$

11. $\log(100) = 2$ and $\ln(\mathrm{e}^{-3}) = -3$ so

$$\log(100) - \ln(\mathrm{e}^{-3}) = 2 - -3 = 5$$

12. (a)
$$\begin{aligned} \mathrm{e}^x + e^{x+1} &= 17 \\ \mathrm{e}^x + e^x \times e^1 &= 17 \\ \mathrm{e}^x(1+\mathrm{e}) &= 17 \\ \mathrm{e}^x &= \frac{17}{1+\mathrm{e}} \\ x &= \ln\frac{17}{1+\mathrm{e}} \\ &= \ln(17) - \ln(1+\mathrm{e}) \end{aligned}$$

(b)
$$\begin{aligned} \mathrm{e}^{2x+1} &= 50^{x-7} \\ 2x + 1 &= \ln(50^{x-7}) \\ &= (x-7)\ln(50) \\ &= \ln(50)x - 7\ln(50) \\ 2x - \ln(50)x &= -7\ln(50) - 1 \\ x\left(2 - \ln(50)\right) &= -(7\ln(50) + 1) \\ x &= \frac{-(7\ln(50)+1)}{2 - \ln(50)} \\ &= \frac{7\ln(50)+1}{\ln(50) - 2} \end{aligned}$$

13.
$$\begin{aligned} P &= 9\mathrm{e}^{(t+1)} \\ \frac{P}{9} &= \mathrm{e}^{(t+1)} \\ \ln\frac{P}{9} &= t + 1 \\ t &= \ln\left(\frac{P}{9}\right) - 1 \end{aligned}$$

(a)
$$\begin{aligned} t &= \ln\left(\frac{180}{9}\right) - 1 \\ &= \ln(20) - 1 \\ &= 1.996 \end{aligned}$$

(b)
$$\begin{aligned} t &= \ln\left(\frac{3600}{9}\right) - 1 \\ &= \ln(400) - 1 \\ &= 4.991 \end{aligned}$$

(c)
$$\begin{aligned} t &= \ln\left(\frac{9e^3}{9}\right) - 1 \\ &= \ln(e^3) - 1 \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

14. (a) Product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x(2x+3) &= 1(2x+3) + 2(x) \\ &= 4x + 3 \end{aligned}$$

(b) Quotient rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{2x+3} &= \frac{1(2x+3) - 2(x)}{(2x+3)^2} \\ &= \frac{3}{(2x+3)^2} \end{aligned}$$

(c) Product and chain rules:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x(2x+3)^4 &= 1(2x+3)^4 + (4(2x+3)^3)(2)(x) \\ &= (2x+3)^4 + 8x(2x+3)^3 \\ &= (2x+3)^3(2x+3+8x) \\ &= (2x+3)^3(10x+3) \end{aligned}$$

15. Examining each piece for solutions:

- First piece:

$$\begin{aligned} -2x &= 1 \qquad x < 0 \\ x &= -\frac{1}{2} \end{aligned}$$

- Second piece:

$$\begin{aligned} x^2 &= 1 \qquad 0 \leq x < 2 \\ x &= \pm 1 \end{aligned}$$

  Discard the negative root as being outside the domain for this piece.

$$x = 1$$

- Third piece:

$$x = 2$$

- Fourth piece:

$$\begin{aligned} x + 2 &= 1 \qquad x > 2 \\ x &= -1 \end{aligned}$$

  No solution in the domain for this piece.

The values of a that satisfy $\mathrm{f}(a) = 1$ are $a = -\frac{1}{2}$, $a = 1$ or $a = 2$.

16.
$$\tan 2x + \tan x = 0$$
$$\tan 2x + \tan x = 0$$
$$\frac{2\tan x}{1-\tan^2 x} + \tan x = 0$$
$$\frac{2\tan x + \tan x(1-\tan^2 x)}{1-\tan^2 x} = 0$$
$$2\tan x + \tan x(1-\tan^2 x) = 0 \quad [1-\tan^2 x \neq 0$$
$$\tan x(2+1-\tan^2 x) = 0$$
$$\tan x(3-\tan^2 x) = 0$$

The first factor gives:
$$\tan x = 0$$
$$x = 0$$
$$\text{or} \quad x = 180°$$
$$\text{or} \quad x = 360°$$

The second factor gives:
$$3-\tan^2 x = 0$$
$$\tan x = \pm\sqrt{3}$$

giving solutions in all four quadrants 60° from the $x-$axis, i.e.
$$x = 60°$$
$$\text{or} \quad x = 120°$$
$$\text{or} \quad x = 240°$$
$$\text{or} \quad x = 300°$$

17. Because **a** and **b** are perpendicular,
$$\mathbf{a}\cdot\mathbf{b} = 0$$
$$(2\mathbf{i}-3\mathbf{j})\cdot(x\mathbf{i}+4\mathbf{j}) = 0$$
$$2x-12 = 0$$
$$x = 6$$

Because **a** and **c** are parallel,
$$\mathbf{c} = k\mathbf{a}$$
$$(9\mathbf{i}-y\mathbf{j}) = k\,(2\mathbf{i}-3\mathbf{j})$$
$$9 = 2k$$
$$k = 4.5$$
$$\text{and} \quad -y = k(-3)$$
$$y = 3k$$
$$= 13.5$$

18. Let $z = a + b\text{i}$ and $w = c + d\text{i}$.
$$\text{Re}(z) = 6$$
$$\therefore \quad a = 6$$
$$\text{Re}(z) + \text{Im}(w) = -3$$
$$6 + d = -3$$
$$d = -9$$
$$3z + 6w = zw$$
$$3(6+b\text{i}) + 6(c-9\text{i}) = (6+b\text{i})(c-9\text{i})$$
$$18 + 3b\text{i} + 6c - 54\text{i} = 6c - 54\text{i} + bc\text{i} - 9b\text{i}^2$$
$$(18+6c) + (3b-54)\text{i} = (6c+9b) + (bc-54)\text{i}$$

equatinging the real components:
$$18 + 6c = 6c + 9b$$
$$18 = 9b$$
$$b = 2$$

equating the imaginary components:
$$3b - 54 = bc - 54$$
$$3b = bc$$
$$c = 3$$
$$\therefore \quad z = 6 + 2\text{i}$$
$$w = 3 - 9\text{i}$$

19. L.H.S.:
$$\sin\theta(\sin\theta + \sin 2\theta)$$
$$= \sin\theta(\sin\theta + 2\sin\theta\cos\theta)$$
$$= \sin^2\theta(1+2\cos\theta)$$
$$= (1-\cos^2\theta)(1+2\cos\theta)$$
$$= 1 + 2\cos\theta - \cos^2\theta - 2\cos^3\theta$$
$$= \text{R.H.S.}$$

□

20.
$$\mathbf{F}+\mathbf{P} = (6\mathbf{i}+4\mathbf{j}) + (2\mathbf{i}-7\mathbf{j})$$
$$= (8\mathbf{i}-3\mathbf{j})\,\text{N}$$
$$|\mathbf{F}| = \sqrt{6^2+4^2}$$
$$= 2\sqrt{13}$$
$$\approx 7.2\text{N}$$
$$|\mathbf{P}| = \sqrt{2^2+(-7)^2}$$
$$= \sqrt{53}$$
$$\approx 7.3\text{N}$$

The angle between the resultant and **F** is given by
$$(6\mathbf{i}+4\mathbf{j})\cdot(8\mathbf{i}-3\mathbf{j}) = |\,(6\mathbf{i}+4\mathbf{j})\,||\,(8\mathbf{i}-3\mathbf{j})\,|\cos\theta$$
$$48 - 12 = 2\sqrt{13}\sqrt{8^2+(-3)^2}\cos\theta$$
$$36 = 2\sqrt{13}\sqrt{73}\cos\theta$$
$$\cos\theta = \frac{36}{2\sqrt{13}\sqrt{73}}$$
$$\theta = \cos^{-1}\frac{18}{\sqrt{13\times 73}}$$
$$= 54° \text{ (nearest degree)}$$

21. L_1 is parallel to $(10\mathbf{i}+4\mathbf{j})$
L_2 is parallel to $(-2\mathbf{i}+5\mathbf{j})$

$$(10\mathbf{i}+4\mathbf{j})\cdot(-2\mathbf{i}+5\mathbf{j}) = 10\times(-2) + 4\times 5$$
$$= 0$$
$$\therefore \quad (10\mathbf{i}+4\mathbf{j}) \perp (-2\mathbf{i}+5\mathbf{j})$$
$$\therefore \quad \text{L}_1 \perp \text{L}_2$$

□

22. (a)
$$R = \sqrt{5^2 + (-3)^2} = \sqrt{34}$$
$$5\cos\theta - 3\sin\theta = \sqrt{34}\left(\frac{5}{\sqrt{34}}\cos\theta - \frac{3}{\sqrt{34}}\sin\theta\right) = \sqrt{34}\,(\cos\alpha\cos\theta - \sin\alpha\sin\theta) = \sqrt{34}\cos(\theta + \alpha)$$
$$\text{where } \cos\alpha = \frac{5}{\sqrt{34}}, \quad \alpha = 0.54$$
$$\therefore \quad 5\cos\theta - 3\sin\theta = \sqrt{34}\cos(\theta + 0.54)$$

(b) The minimum value is $-\sqrt{34}$ and this occurs when
$$\cos(\theta + 0.54) = -1$$
$$\theta + 0.54 = \pi$$
$$\theta = \pi - 0.54 = 2.60 \text{ (2 d.p.)}$$

23. (a) Each piece of the function is individually continuous so it only remains to consider the continuity at $x = 1$ and at $x = 3$.
The function is discontinuous at $x = 1$ because f(1) is not defined.
The function is continuous at $x = 3$ because
$$\lim_{x\to 3^-} \mathrm{f}(x) = 2(3) - 1 = 5$$
$$\lim_{x\to 3^+} \mathrm{f}(x) = f(3) = 4(3) - 7 = 5$$
$\mathrm{f}(x)$ is continuous $\forall x \in \Re, x \neq 1$.

(b) i. From the left,
$$\lim_{x\to 0^-} \mathrm{f}'(x) = -1$$
but from the right
$$\lim_{x\to 0^+} \mathrm{f}'(x) = 1$$
so $\mathrm{f}(x)$ is not differentiable at $x = 0$.

ii. $\mathrm{f}(x)$ is differentiable at $x = 0.5$ as it is continuous and the gradient of the curve for all $0 < x < 1$ is 1 so the derivative from the left is equal to the derivative from the right.

iii. $\mathrm{f}(x)$ is not continuous and therefore not differentiable at $x = 1$.

iv. From the left,
$$\lim_{x\to 3^-} \mathrm{f}'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(2x - 1) = 2$$
and from the right
$$\lim_{x\to 3^+} \mathrm{f}'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(4x - 7) = 4$$
so $\mathrm{f}(x)$ is not differentiable at $x = 3$.

24. Let P be the point on the line nearest A.
$$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\begin{pmatrix}-12\\-10\end{pmatrix} + \begin{pmatrix}-6\\12\end{pmatrix} + \lambda\begin{pmatrix}5\\1\end{pmatrix} = \begin{pmatrix}6\\22\end{pmatrix} + \lambda\begin{pmatrix}5\\1\end{pmatrix}$$

$\overrightarrow{AP}$ is perpendicular to the line so
$$\overrightarrow{AP}\cdot\begin{pmatrix}5\\1\end{pmatrix} = 0$$
$$\left(\begin{pmatrix}6\\22\end{pmatrix} + \lambda\begin{pmatrix}5\\1\end{pmatrix}\right)\cdot\begin{pmatrix}5\\1\end{pmatrix} = 0$$
$$\begin{pmatrix}6\\22\end{pmatrix}\cdot\begin{pmatrix}5\\1\end{pmatrix} + \lambda\begin{pmatrix}5\\1\end{pmatrix}\cdot\begin{pmatrix}5\\1\end{pmatrix} = 0$$
$$(30 + 22) + \lambda(25 + 1) = 0$$
$$52 + 26\lambda = 0$$
$$\lambda = -2$$

$$\overrightarrow{AP} = \begin{pmatrix}6\\22\end{pmatrix} - 2\begin{pmatrix}5\\1\end{pmatrix}$$
$$\overrightarrow{AP} = \begin{pmatrix}-4\\20\end{pmatrix}$$
$$|\overrightarrow{AP}| = \sqrt{(-4)^2 + 20^2} = 4\sqrt{26}$$

25. After 5 seconds
$$v = \frac{200}{3}\left(1 - \mathrm{e}^{-0.15\times 5}\right) = 35.2\mathrm{ms}^{-1}$$
Terminal velocity is given by
$$\lim t \to \infty\left(\frac{200}{3}\left(1 - \mathrm{e}^{-0.15t}\right)\right) = \frac{200}{3}(1 - 0) = \frac{200}{3}\mathrm{ms}^{-1}$$
$$(\approx 66.7\mathrm{ms}^{-1})$$

26.
$$x^2 + 2x + y^2 - 10y + a = 0$$
$$(x + 1)^2 - 1 + (y - 5)^2 - 25 + a = 0$$
$$(x + 1)^2 + (y - 5)^2 = 26 - a$$
If this is the equation of a circle then it must have a real, positive radius, so $26 - a > 0$ or $a < 26$.

27. (a)

$$\begin{aligned}|(-10\mathbf{i}+24\mathbf{j}+\lambda(5\mathbf{i}+\mathbf{j}))-(34\mathbf{i}+12\mathbf{j})| &= 2\sqrt{130}\\ |-44\mathbf{i}+12\mathbf{j}+\lambda(5\mathbf{i}+\mathbf{j})| &= 2\sqrt{130}\\ |(5\lambda-44)\mathbf{i}+(\lambda+12)\mathbf{j}| &= 2\sqrt{130}\\ (5\lambda-44)^2+(\lambda+12)^2 &= (2\sqrt{130})^2\\ 25\lambda^2-440\lambda+1936 & \\ +\lambda^2+24\lambda+144 &= 520\\ 26\lambda^2-416\lambda+1560 &= 0\\ \lambda^2-16\lambda+60 &= 0\\ (\lambda-6)(\lambda-10) &= 0\end{aligned}$$

$$\begin{aligned}\lambda &= 6\\ \mathbf{r} &= -10\mathbf{i}+24\mathbf{j}+6(5\mathbf{i}+\mathbf{j})\\ &= 20\mathbf{i}+30\mathbf{j}\\ \text{or}\quad \lambda &= 10\\ \mathbf{r} &= -10\mathbf{i}+24\mathbf{j}+10(5\mathbf{i}+\mathbf{j})\\ &= 40\mathbf{i}+34\mathbf{j}\end{aligned}$$

The line intersects the circle at two points: $(20\mathbf{i}+30\mathbf{j})$ and $(40\mathbf{i}+34\mathbf{j})$.

(b)

$$\begin{aligned}|(-\mathbf{i}+5\mathbf{j}+\lambda(3\mathbf{i}-\mathbf{j}))-(3\mathbf{i}+\mathbf{j})| &= \sqrt{5}\\ |-4\mathbf{i}+4\mathbf{j}+\lambda(3\mathbf{i}-\mathbf{j})| &= \sqrt{5}\\ |(3\lambda-4)\mathbf{i}+(4-\lambda)\mathbf{j}| &= \sqrt{5}\\ (3\lambda-4)^2+(4-\lambda)^2 &= 5\\ 9\lambda^2-24\lambda+16+16-8\lambda+\lambda^2 &= 5\\ 10\lambda^2-32\lambda+32 &= 5\\ 10\lambda^2-32\lambda+27 &= 0\end{aligned}$$

$$\begin{aligned}\lambda &= \frac{32\pm\sqrt{(-32)^2-4\times 10\times 27}}{2\times 10}\\ &= \frac{32\pm\sqrt{1024-1080}}{20}\end{aligned}$$

Since the discriminant is negative it is clear that this has no real solutions. The line does not intersect the circle.

(c)

$$\begin{aligned}|(-\mathbf{i}+7\mathbf{j}+\lambda(\mathbf{i}+3\mathbf{j}))-(4\mathbf{i}+2\mathbf{j})| &= 2\sqrt{10}\\ |-5\mathbf{i}+5\mathbf{j}+\lambda(\mathbf{i}+3\mathbf{j})| &= 2\sqrt{10}\\ |(\lambda-5)\mathbf{i}+(3\lambda+5)\mathbf{j}| &= 2\sqrt{10}\\ (\lambda-5)^2+(3\lambda+5)^2 &= (2\sqrt{10})^2\\ \lambda^2-10\lambda+25+9\lambda^2+30\lambda+25 &= 40\\ 10\lambda^2+20\lambda+10 &= 0\\ \lambda^2+2\lambda+1 &= 0\\ (\lambda+1)^2 &= 0\end{aligned}$$

$$\begin{aligned}\lambda &= -1\\ \mathbf{r} &= -\mathbf{i}+7\mathbf{j}-(\mathbf{i}+3\mathbf{j})\\ &= -2\mathbf{i}+4\mathbf{j}\end{aligned}$$

The line intersects the circle at a single point: $(-2\mathbf{i}+4\mathbf{j})$ (that is, the line is a tangent to the circle).