Chapter 7

Exercise 7A

1. I will use the "intelligent guess" method for this question, but my preference is for the "rearranging" method, so I will use that for most of the questions where one of these approaches is suitable.
   Guess $y = \frac{(2x+5)^5}{5}$
   $\frac{\mathrm{d}}{\mathrm{d}x}\frac{(2x+5)^5}{5} = (2x+5)^4(2)$
   Adjust by a factor of $\frac{1}{2}$:
   $$\frac{(2x+5)^5}{10} + c$$

2. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (3x+1)^3\\ &= \frac{1}{3}(3(3x+1)^3)\\ y &= \frac{1}{3}\frac{(3x+1)^4}{4} + c\\ &= \frac{(3x+1)^4}{12} + c\end{aligned}$$

3. This can not be rearranged to the form $\mathrm{f}'(x)(\mathrm{f}(x))^n$ so we expand:
   $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x(3x+4)\\ &= 3x^2 + 4x\\ y &= \frac{3x^3}{3} + \frac{4x^2}{2} + c\\ &= x^3 + 2x^2 + c\end{aligned}$$

4. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 50(1+5x)^4\\ &= 10(5(1+5x)^4)\\ y &= 10\left(\frac{(1+5x)^5}{5}\right) + c\\ &= 2(1+5x)^5 + c\end{aligned}$$

5. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 24x(2-x^2)^3\\ &= -12(-2x(2-x^2)^3)\\ y &= -12\left(\frac{(2-x^2)^4}{4}\right) + c\\ &= -3(2-x^2)^4 + c\end{aligned}$$

6. This can not be rearranged to the form $\mathrm{f}'(x)(\mathrm{f}(x))^n$ so we expand:
   $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x(1+4x)^2\\ &= x(1+8x+16x^2)\\ &= x + 8x^2 + 16x^3\\ y &= \frac{x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + c\\ &= \frac{x^2}{2} + \frac{8x^3}{3} + 4x^4 + c\end{aligned}$$

7. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 30x(x^2-3)^2\\ &= 15(2x(x^2-3)^2)\\ y &= 15\left(\frac{(x^2-3)^3}{3}\right) + c\\ &= 5(x^2-3)^3 + c\end{aligned}$$

8. This can not be rearranged to the form $\mathrm{f}'(x)(\mathrm{f}(x))^n$ so we expand:
   $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x(2x-3)^2\\ &= x(4x^2-12x+9)\\ &= 4x^3 - 12x^2 + 9x\\ y &= \frac{4x^4}{4} - \frac{12x^3}{3} + \frac{9x^2}{2} + c\\ &= x^4 - 4x^3 + \frac{9x^2}{2} + c\end{aligned}$$

9. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 12(2x+1)^3\\ &= 6(2(2x+1)^3)\\ y &= 6\left(\frac{(2x+1)^4}{4}\right) + c\\ &= \frac{3(2x+1)^4}{2} + c\end{aligned}$$

10. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2(3x+1)^4\\ &= \frac{2}{3}(3(3x+1)^4)\\ y &= \frac{2}{3}\left(\frac{(3x+1)^5}{5}\right) + c\\ &= \frac{2(3x+1)^5}{15} + c\end{aligned}$$

11. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (2x-3)^4\\ &= \frac{1}{2}(2(2x-3)^4)\\ y &= \frac{1}{2}\left(\frac{(2x-3)^5}{5}\right) + c\\ &= \frac{(2x-3)^5}{10} + c\end{aligned}$$

12. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 5x^2(3-x^3)^4\\ &= \frac{5}{-3}(-3x^2(3-x^3)^4)\\ y &= -\frac{5}{3}\left(\frac{(3-x^3)^5}{5}\right) + c\\ &= -\frac{(3-x^3)^5}{3} + c\\ &= \frac{(x^3-3)^5}{3} + c\end{aligned}$$
    (Note that the last step is valid because $(-1)^5 =$

−1. Such a simplification would not be possible if we were raising to an even power.)

13. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (1+x)^4\\ y &= \frac{(1+x)^5}{5}+c\end{aligned}$$

14. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x(2+x^2)^4\\ &= 2(2x(2+x^2)^4)\\ y &= 2\frac{(2+x^2)^5}{5}+c\\ &= \frac{2(2+x^2)^5}{5}+c\end{aligned}$$

15. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (1+2x)^4\\ &= \frac{1}{2}(2(1+2x)^4)\\ y &= \frac{1}{2}\left(\frac{(1+2x)^5}{5}\right)+c\\ &= \frac{(1+2x)^5}{10}+c\end{aligned}$$

16. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x(1+x^2)\\ &= 2(2x(1+x^2))\\ y &= 2\left(\frac{(1+x^2)^2}{2}\right)+c\\ &= (1+x^2)^2+c\end{aligned}$$

This question is probably just as easy to do by first expanding:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x(1+x^2)\\ &= 4x+4x^3\\ y &= 4\frac{x^2}{2}+4\frac{x^4}{3}+c\\ &= 2x^2+x^4+c\end{aligned}$$

Although these two solutions may not look quite the same, you should satisfy yourself that they are, in fact, both correct. (Remember, c is an *arbitrary* constant.)

17. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 60(2x-3)^5\\ &= 30(2(2x-3)^5\\ y &= 30\left(\frac{(2x-3)^6}{6}\right)+c\\ &= 5(2x-3)^6+c\end{aligned}$$

18. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 60(3-2x)^5\\ &= -30(-2(3-2x)^5\\ y &= -30\left(\frac{(3-2x)^6}{6}\right)+c\\ &= -5(3-2x)^6+c\end{aligned}$$

19. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{(x+2)^4}\\ &= (x+2)^{-4}\\ y &= \frac{(x+2)^{-3}}{-3}+c\\ &= -\frac{1}{3(x+2)^3}+c\end{aligned}$$

20. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{(2x+1)^4}\\ &= (2x+1)^{-4}\\ &= \frac{1}{2}(2(2x+1)^{-4}\\ y &= \frac{1}{2}\frac{(2x+1)^{-3}}{-3}+c\\ &= -\frac{1}{6(2x+1)^3}+c\end{aligned}$$

21. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{25x}{(x^2+1)^5}\\ &= -25x(x^2+1)^{-5}\\ &= \frac{-25}{2}(2x(x^2+1)^{-5})\\ y &= \frac{-25}{2}\frac{(x^2+1)^{-4}}{-4}+c\\ &= \frac{25}{8(x^2+1)^4}+c\end{aligned}$$

22. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6}{(3x+5)^3}\\ &= 6(3x+5)^{-3}\\ &= 2(3(3x+5)^{-3})\\ y &= 2\left(\frac{(3x+5)^{-2}}{-2}\right)+c\\ &= -\frac{1}{(3x+5)^2}+c\end{aligned}$$

23. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{18x}{(3x^2+5)^3}\\ &= 3(6x)(3x^2+5)^{-3}\\ y &= 3\left(\frac{(3x^2+5)^{-2}}{-2}\right)+c\\ &= -\frac{3}{2(3x^2+5)^2}+c\end{aligned}$$

24. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 12\sqrt[3]{3x-2}\\ &= 4\left(3(3x-2)^{\frac{1}{3}}\right)\\ y &= 4\left(\frac{(3x-2)^{\frac{4}{3}}}{\frac{4}{3}}\right)+c\\ &= 4(3x-2)^{\frac{4}{3}}\left(\frac{3}{4}\right)+c\\ &= 3(3x-2)^{\frac{4}{3}}+c\end{aligned}$$

25. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 12\sqrt{3x+5}\\ &= 4\left(3(3x+5)^{\frac{1}{2}}\right)\\ y &= 4\left(\frac{(3x+5)^{\frac{3}{2}}}{\frac{3}{2}}\right)+c\\ &= 4(3x+5)^{\frac{3}{2}}\left(\frac{2}{3}\right)+c\\ &= \frac{8(3x+5)^{\frac{3}{2}}}{3}+c\end{aligned}$$

26. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{12}{\sqrt{3x+5}}\\ &= 4\left(3(3x+5)^{-\frac{1}{2}}\right)\\ y &= 4\left(\frac{(3x+5)^{\frac{1}{2}}}{\frac{1}{2}}\right)+c\\ &= 8(3x+5)^{\frac{1}{2}}+c\\ &= 8\sqrt{3x+5}+c\end{aligned}$$

27. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3-12(-3x^2(1-x^3)^2)\\ y &= 3x-12\left(\frac{(1-x^3)^3}{3}\right)+c\\ &= 3x-4(1-x^3)^3+c\end{aligned}$$

28. $$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= 12(3(2+3t)^3)\\ P &= 12\left(\frac{(2+3t)^4}{4}\right)+c\\ &= 3(2+3t)^4+c\\ 50 &= 3(2+3(0))^4+c\\ 50 &= 3(2^4)+c\\ c &= 50-3\times 16\\ &= 2\\ \therefore\quad P &= 3(2+3t)^4+2\end{aligned}$$

29. $$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= 12(2t(t^2-5)^3\\ P &= 12\left(\frac{(t^2-5)^4}{4}\right)+c\\ &= 3(t^2-5)^4+c\\ 10 &= 3(2^2-5)^4+c\\ 10 &= 3(-1)^4+c\\ 10 &= 3+c\\ c &= 7\\ \therefore\quad P &= 3(t^2-5)^4+7\end{aligned}$$

Exercise 7B

1. $$\int 5\cos x\,\mathrm{d}x = 5\sin x + c$$

2. $$\int 2\sin x\,\mathrm{d}x = -2\cos x + c$$

3. $$\int -10\sin x\,\mathrm{d}x = 10\cos x + c$$

4. $$\int -2\cos x\,\mathrm{d}x = -2\sin x + c$$

5. $$\begin{aligned}\int 6\cos 2x\,\mathrm{d}x &= 3\int 2\cos 2x\,\mathrm{d}x\\ &= 3\sin 2x + c\end{aligned}$$

6. $$\begin{aligned}\int 2\cos 6x\,\mathrm{d}x &= \frac{1}{3}\int 6\cos 6x\,\mathrm{d}x\\ &= \frac{\sin 6x}{3}+c\end{aligned}$$

7. $$\begin{aligned}\int 12\sin 4x\,\mathrm{d}x &= -3\int -4\sin 4x\,\mathrm{d}x\\ &= -3\cos 4x + c\end{aligned}$$

8. $$\begin{aligned}\int -\sin 3x\,\mathrm{d}x &= \frac{1}{3}\int -3\sin 3x\,\mathrm{d}x\\ &= \frac{\cos 3x}{3}+c\end{aligned}$$

9. $$\begin{aligned}\int -8\cos 10x\,\mathrm{d}x &= -\frac{4}{5}\int 10\cos 10x\,\mathrm{d}x\\ &= -\frac{4\sin 10x}{5}+c\end{aligned}$$

10. $$\begin{aligned}\int \sin\frac{x}{2}\,\mathrm{d}x &= -2\int -\frac{1}{2}\sin\frac{x}{2}\,\mathrm{d}x\\ &= -2\cos\frac{x}{2}+c\end{aligned}$$

11. $$\begin{aligned}\int \cos\frac{3x}{2}\,\mathrm{d}x &= \frac{2}{3}\int \frac{3}{2}\cos\frac{3x}{2}\,\mathrm{d}x\\ &= \frac{2}{3}\sin\frac{3x}{2}+c\end{aligned}$$

12. $$\begin{aligned}\int -6\sin\frac{2x}{3}\,\mathrm{d}x &= \left(6\times\frac{3}{2}\right)\int -\frac{2}{3}\sin\frac{2x}{3}\,\mathrm{d}x\\ &= 9\cos\frac{2x}{3}+c\end{aligned}$$

13. For this question and the next, you should note that $\sin\left(x+\frac{\pi}{2}\right)=\cos x$ and $\cos\left(x-\frac{\pi}{2}\right)=\sin x$. You may make the substitution at the beginning and then antidifferentiate, or first antidifferentiate and then substitute.

14. See previous question.

15. $$\int \cos\left(2x+\frac{2\pi}{3}\right)\mathrm{d}x = \frac{1}{2}\int 2\cos\left(2x+\frac{2\pi}{3}\right)\mathrm{d}x = \sin\left(2x+\frac{2\pi}{3}\right)+c$$

16. $$\int \sin(-x)\,\mathrm{d}x = \int -\sin x\,\mathrm{d}x = \cos x + c$$

17. $$\int \frac{4}{\cos^2 x}\,\mathrm{d}x = 4\tan x + c$$

18. $$\int \frac{1}{\cos^2 4x}\,\mathrm{d}x = \frac{1}{4}\int\frac{4}{\cos^2 4x}\,\mathrm{d}x = \frac{1}{4}\tan(4x)+c$$

19. $$\int \frac{1}{\cos^2(x-1)}\,\mathrm{d}x = \tan(x-1)+c$$

20. $$\int 6\cos 2x + 6\sin 3x\,\mathrm{d}x = 3\int 2\cos 2x\,\mathrm{d}x - 2\int -3\sin 3x\,\mathrm{d}x = 3\sin 2x - 2\cos 3x + c$$

21. $$\int \cos 8x - 4\sin 2x\,\mathrm{d}x = \frac{1}{8}\int 8\cos 8x\,\mathrm{d}x + 2\int -2\sin 2x\,\mathrm{d}x = \frac{1}{8}\sin 8x + 2\cos 2x + c$$

22. $$\int 2x + 4\cos x + 6\cos 2x\,\mathrm{d}x = \int 2x\,\mathrm{d}x + 4\int\cos x\,\mathrm{d}x + 3\int 2\cos 2x\,\mathrm{d}x = x^2 + 4\sin x + 3\sin 2x + c$$

23. Although this looks long, you should by now be able to antidifferentiate it in a single step, simply working term by term.
$$\int 3+4x-6x^2\,\mathrm{d}x = 3x+2x^2-2x^3+c_1$$
$$\int 10\cos 5x\,\mathrm{d}x = 2\int 5\cos 5x\,\mathrm{d}x = 2\sin 5x + c_2$$
$$\int -2\sin 4x\,\mathrm{d}x = \frac{1}{2}\int -4\sin 4x\,\mathrm{d}x = \frac{\cos 4x}{2}+c_3$$
so the overall antiderivative is
$$3x+2x^2-2x^3+2\sin 5x+\frac{\cos 4x}{2}+c$$

24. $$\int \cos^3 x\sin x\,\mathrm{d}x = -\frac{\cos^4 x}{4}+c$$

25. $$\int 30\cos^5 x\sin x\,\mathrm{d}x = -\frac{30\cos^6 x}{6}+c = -5\cos^6 x + c$$

26. $$\int \sin^4 x\cos x\,\mathrm{d}x = \frac{\sin^5 x}{5}+c$$

27. $$\int 6\sin^3 x\cos x\,\mathrm{d}x = \frac{6\sin^4 x}{4}+c = \frac{3\sin^4 x}{2}+c$$

28. $$\int -2\cos^4 x\sin x\,\mathrm{d}x = \frac{2\cos^5 x}{5}+c$$

29. $$\int -2\sin^7 x\cos x\,\mathrm{d}x = -\frac{2\sin^8 x}{8}+c = -\frac{\sin^8 x}{4}+c$$

30. $$\int 32\sin^3 4x\cos 4x\,\mathrm{d}x = 8\int \sin^3 4x(4\cos 4x)\,\mathrm{d}x = \frac{8\sin^4 4x}{4}+c = 2\sin^4 4x + c$$

31. $$\int -24\sin^3 2x\cos 2x\,\mathrm{d}x = -12\int\sin^3 2x(2\cos 2x)\,\mathrm{d}x = \frac{-12\sin^4 2x}{4}+c = -3\sin^4 2x + c$$

32. $$\int 20\sin^4 2x\cos 2x\,\mathrm{d}x = 10\int \sin^4 2x(2\cos 2x)\,\mathrm{d}x = \frac{10\sin^5 2x}{5}+c = 2\sin^5 2x + c$$

33. $$\int -6\cos^2 4x\sin 4x\,\mathrm{d}x = \frac{3}{2}\int\cos^2 4x(-4\sin 4x)\,\mathrm{d}x = \frac{3\cos^3 4x}{2\times 3}+c = \frac{\cos^3 4x}{2}+c$$

34. $$\begin{aligned}\int \sin^3 x\,\mathrm{d}x &= \int \sin x(\sin^2 x)\,\mathrm{d}x\\ &= \int \sin x(1-\cos^2 x)\,\mathrm{d}x\\ &= \int \sin x - \sin x\cos^2 x\,\mathrm{d}x\\ &= -\cos x + \frac{\cos^3 x}{3} + c\end{aligned}$$

35. $$\begin{aligned}\int \cos^3 x\,\mathrm{d}x &= \int \cos x(\cos^2 x)\,\mathrm{d}x\\ &= \int \cos x(1-\sin^2 x)\,\mathrm{d}x\\ &= \int \cos x - \cos x\sin^2 x\,\mathrm{d}x\\ &= \sin x - \frac{\sin^3 x}{3} + c\end{aligned}$$

36. $$\begin{aligned}&\int \cos^5 x\,\mathrm{d}x\\ &= \int \cos x(\cos^4 x)\,\mathrm{d}x\\ &= \int \cos x(\cos^2 x)^2\,\mathrm{d}x\\ &= \int \cos x(1-\sin^2 x)^2\,\mathrm{d}x\\ &= \int \cos x(1-2\sin^2 x+\sin^4 x)\,\mathrm{d}x\\ &= \int \cos x - 2\cos x\sin^2 x + \cos x\sin^4 x\,\mathrm{d}x\\ &= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c\end{aligned}$$

37. $$\begin{aligned}\int \cos^2 x\,\mathrm{d}x &= \frac{1}{2}\int 2\cos^2 x\,\mathrm{d}x\\ &= \frac{1}{2}\int 2\cos^2 x - 1 + 1\,\mathrm{d}x\\ &= \frac{1}{2}\int \cos 2x + 1\,\mathrm{d}x\\ &= \frac{1}{2}\left(\frac{\sin 2x}{2} + x\right) + c\\ &= \frac{\sin 2x}{4} + \frac{x}{2} + c\end{aligned}$$

38. $$\begin{aligned}\int \sin^2 x\,\mathrm{d}x &= -\frac{1}{2}\int -2\sin^2 x\,\mathrm{d}x\\ &= -\frac{1}{2}\int 1 - 2\sin^2 x - 1\,\mathrm{d}x\\ &= -\frac{1}{2}\int \cos 2x - 1\,\mathrm{d}x\\ &= -\frac{1}{2}\left(\frac{\sin 2x}{2} - x\right) + c\\ &= -\frac{\sin 2x}{4} + \frac{x}{2} + c\end{aligned}$$

39. $$\begin{aligned}\int 8\sin^4 x\,\mathrm{d}x &= 8\int (\sin^2 x)^2\,\mathrm{d}x\\ &= 8\int \left(\frac{-2\sin^2 x}{-2}\right)^2\,\mathrm{d}x\\ &= 8\int \frac{(-2\sin^2 x)^2}{4}\,\mathrm{d}x\\ &= 2\int (-2\sin^2 x)^2\,\mathrm{d}x\\ &= 2\int (1-2\sin^2 x-1)^2\,\mathrm{d}x\\ &= 2\int (\cos 2x - 1)^2\,\mathrm{d}x\\ &= 2\int \cos^2 2x - 2\cos 2x + 1\,\mathrm{d}x\\ &= \int 2\cos^2 2x - 4\cos 2x + 2\,\mathrm{d}x\\ &= \int 2\cos^2 2x - 1 - 4\cos 2x + 3\,\mathrm{d}x\\ &= \int \cos 4x - 4\cos 2x + 3\,\mathrm{d}x\\ &= \frac{\sin 4x}{4} - \frac{4\sin 2x}{2} + 3x + c\\ &= \frac{\sin 4x}{4} - 2\sin 2x + 3x + c\end{aligned}$$

40. $$\begin{aligned}\int \cos^2 x + \sin^2 x\,\mathrm{d}x &= \int 1\,\mathrm{d}x\\ &= x + c\end{aligned}$$

(Compare this with your answers for questions 37 and 38.)

41. $$\begin{aligned}\int \cos^2 x - \sin^2 x\,\mathrm{d}x &= \int \cos 2x\,\mathrm{d}x\\ &= \frac{\sin 2x}{2} + c\end{aligned}$$

(Compare this with your answers for questions 37 and 38.)

42. $$\begin{aligned}&\int \sin 5x\cos 2x + \cos 5x\sin 2x\,\mathrm{d}x\\ &= \int \sin(5x+2x)\,\mathrm{d}x\\ &= \int \sin 7x\,\mathrm{d}x\\ &= -\frac{\cos 7x}{7} + c\end{aligned}$$

(You need to know the angle sum trig identities well enough to recognise them.)

43. $$\begin{aligned}&\int \sin 3x\cos x - \cos 3x\sin x\,\mathrm{d}x\\ &= \int \sin(3x-x)\,\mathrm{d}x\\ &= \int \sin 2x\,\mathrm{d}x\\ &= -\frac{\cos 2x}{2} + c\end{aligned}$$

44. $\int \cos 5x \cos 2x - \sin 5x \sin 2x \,\mathrm{d}x$

$$= \int \cos(5x + 2x) \,\mathrm{d}x$$
$$= \int \cos 7x \,\mathrm{d}x$$
$$= \frac{\sin 7x}{7} + c$$

45. $\int \cos 5x \cos x + \sin 5x \sin x \,\mathrm{d}x$

$$= \int \cos(5x - x) \,\mathrm{d}x$$
$$= \int \cos 4x \,\mathrm{d}x$$
$$= \frac{\sin 4x}{4} + c$$

46. Refer to your answers to questions 34 and 37.

47. There are (at least) three different ways of approaching this question. You can treat it as
    - $\int 2f(x)f'(x)\,\mathrm{d}x = [f(x)]^2 + c$ where $f(x) = \sin x$ (i.e. $sin^2 x + c$);
    - $\int -2f(x)f'(x)\,\mathrm{d}x = -[f(x)]^2 + c$ where $f(x) = \cos x$ (i.e. $-cos^2 x + c$); or
    - Recognise the double angle formula for sine: $\int \sin 2x \,\mathrm{d}x = -\frac{1}{2}\cos 2x + c$.

    You should satisfy yourself that these answers are equivalent (differing only in the value of the constant of integration).

48. $\int \sin^3 x \cos^2 x \,\mathrm{d}x$

$$= \int \sin x \sin^2 x \cos^2 x \,\mathrm{d}x$$
$$= \int \sin x (1 - \cos^2 x) \cos^2 x \,\mathrm{d}x$$
$$= \int \sin x \cos^2 x - \sin x \cos^4 x \,\mathrm{d}x$$
$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

49. $\int \cos^3 x \sin^2 x \,\mathrm{d}x$

$$= \int \cos x \cos^2 x \sin^2 x \,\mathrm{d}x$$
$$= \int \cos x (1 - \sin^2 x) \sin^2 x \,\mathrm{d}x$$
$$= \int \cos x \sin^2 x - \cos x \sin^4 x \,\mathrm{d}x$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

50.
$$\begin{aligned} C &= \int 4 \sin 2p \, dp \\ &= -2\cos 2p + k \\ 3 &= -2\cos 0 + k \\ k &= 5 \\ \therefore \quad C &= 5 - 2\cos 2p \end{aligned}$$

Note that I use k for the constant of integration here rather than the more commonly used c so as to avoid confusion with C.

51.
$$\begin{aligned} P &= \int 12 \cos 3x \,\mathrm{d}x \\ &= 4 \sin 3x + c \\ 10 &= 4 \sin \frac{3\pi}{2} + c \\ 10 &= -4 + c \\ c &= 14 \\ \therefore \quad P &= 4 \sin 3x + 14 \end{aligned}$$

52. $\int 2 \sec x \tan x \,\mathrm{d}x = 2 \sec x + c$

53. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec 2x) = 2 \sec 2x \tan 2x$$

so

$$\int \sec 2x \tan 2x \,\mathrm{d}x = \frac{\sec 2x}{2} + c$$

54. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot 2x) = -2 \operatorname{cosec}^2 2x$$

so

$$\int -4 \operatorname{cosec}^2 2x \,\mathrm{d}x = 2 \cot 2x + c$$

55.
$$\begin{aligned} \int \frac{1}{\sin^2 x} \,\mathrm{d}x &= \int \operatorname{cosec}^2 x \,\mathrm{d}x \\ &= -\cot x + c \end{aligned}$$

56.
$$\begin{aligned} \int \frac{\sin x}{\cos^2 x} \,\mathrm{d}x &= \int \frac{1}{\cos x} \frac{\sin x}{cos x} \,\mathrm{d}x \\ &= \int \sec x \tan x \,\mathrm{d}x \\ &= \sec x + c \end{aligned}$$

57.
$$\begin{aligned} \int \frac{\cos x}{\sin^2 x} \,\mathrm{d}x &= \int \operatorname{cosec} x \cot x \,\mathrm{d}x \\ &= -\operatorname{cosec} x + c \end{aligned}$$

58. This is of the form $\int f(x)^n f'(x)\,\mathrm{d}x$ where $f(x) = \operatorname{cosec} x$:

$$\begin{aligned} \int 20(\operatorname{cosec} x \cot x)(\operatorname{cosec}^3 x)\,\mathrm{d}x &= \frac{-20 \operatorname{cosec}^4 x}{4} + c \\ &= -5 \operatorname{cosec}^4 x + c \end{aligned}$$

59. $\int 20 \operatorname{cosec}^4 x \cot x \,\mathrm{d}x$

$$= \int 20(\operatorname{cosec} x \cot x)(\operatorname{cosec}^3 x)\,\mathrm{d}x$$
$$= -5 \operatorname{cosec}^4 x + c$$

Exercise 7C

In the following solutions I have used notation like $du = 2x\,\mathrm{d}x$. This is convenient in helping us make the variable substitutions, but it is important to remember that du and dx are meaningless outside the context of integration. We must not fall into the trap of treating them like real quantities.

1. $u = x^2 - 3 \quad du = 2x\,\mathrm{d}x$

$$\begin{aligned}\int 60x(x^2-3)^5\,\mathrm{d}x &= 30\int u^5\,2x\,\mathrm{d}x\\ &= 30\int u^5\,du\\ &= 5u^6 + c\\ &= 5(x^2-3)^6 + c\end{aligned}$$

2. $u = 1 - 2x \quad du = -2\,\mathrm{d}x$

$x = \dfrac{1-u}{2} \quad dx = -\dfrac{1}{2}\,du$

$$\begin{aligned}&\int 80x(1-2x)^3\,\mathrm{d}x\\ &= \int 80\left(\frac{1-u}{2}\right)u^3\left(-\frac{1}{2}\right)\,du\\ &= \int -20(1-u)u^3\,du\\ &= 20\int u^4 - u^3\,du\\ &= 20\left(\frac{u^5}{5} - \frac{u^4}{4}\right) + c\\ &= 4u^5 - 5u^4 + c\\ &= u^4(4u-5) + c\\ &= (1-2x)^4\,(4(1-2x)-5) + c\\ &= (1-2x)^4(4-8x-5) + c\\ &= -(1-2x)^4(8x+1) + c\end{aligned}$$

3. $u = 3x + 1 \quad du = 3\,\mathrm{d}x$

$x = \dfrac{u-1}{3} \quad dx = \dfrac{1}{3}\,du$

$$\begin{aligned}&\int 12x(3x+1)^5\,\mathrm{d}x\\ &= \int 12\left(\frac{u-1}{3}\right)u^5\left(\frac{1}{3}\right)\,du\\ &= \int \frac{4}{3}(u-1)u^5\,du\\ &= \frac{4}{3}\int u^6 - u^5\,du\\ &= \frac{4}{3}\left(\frac{u^7}{7} - \frac{u^6}{6}\right) + c\\ &= \frac{4u^7}{21} - \frac{2u^6}{9} + c\\ &= \frac{2u^6}{63}(6u-7) + c\\ &= \frac{2(3x+1)^6}{63}\,(6(3x+1)-7) + c\\ &= \frac{2(3x+1)^6}{63}(18x+6-7) + c\\ &= \frac{2}{63}(3x+1)^6(18x-1) + c\end{aligned}$$

4. $u = 2x^2 - 1 \quad du = 4x\,\mathrm{d}x$

$$\begin{aligned}\int 6x(2x^2-1)^5\,\mathrm{d}x &= \int \frac{3}{2}u^5 4x\,\mathrm{d}x\\ &= \frac{3}{2}\int u^5\,du\\ &= \frac{3}{2}\frac{u^6}{6} + c\\ &= \frac{1}{4}u^6 + c\\ &= \frac{1}{4}(2x^2-1)^6 + c\end{aligned}$$

5. $u = 3x^2 + 1 \quad du = 6x\,\mathrm{d}x$

$$\begin{aligned}\int 12x(3x^2+1)^5\,\mathrm{d}x &= \int 2u^5 6x\,\mathrm{d}x\\ &= 2\int u^5\,du\\ &= 2\frac{u^6}{6} + c\\ &= \frac{1}{3}u^6 + c\\ &= \frac{1}{3}(3x^2+1)^6 + c\end{aligned}$$

6. $u = x - 2 \quad du = dx$

$x = u + 2 \quad dx = du$

$$\begin{aligned}\int 3x(x-2)^5\,\mathrm{d}x &= \int 3(u+2)u^5\,du\\ &= \int 3u^6+6u^5\,du\\ &= 3\frac{u^7}{7}+6\frac{u^6}{6}+c\\ &= 3\frac{u^7}{7}+u^6+c\\ &= \frac{1}{7}u^6(3u+7)+c\\ &= \frac{1}{7}(x-2)^6\left(3(x-2)+7\right)+c\\ &= \frac{1}{7}(x-2)^6(3x-6+7)+c\\ &= \frac{1}{7}(x-2)^6(3x+1)+c\end{aligned}$$

7. $u=3-x \quad du=-dx$
   $x=3-u \quad dx=-du$

$$\begin{aligned}\int 20x(3-x)^3\,\mathrm{d}x &= \int -20(3-u)u^3\,du\\ &= 20\int u^4-3u^3\,du\\ &= 20\left(\frac{u^5}{5}-\frac{3u^4}{4}\right)+c\\ &= 4u^5-15u^4+c\\ &= u^4(4u-15)+c\\ &= (3-x)^4\left(4(3-x)-15\right)+c\\ &= (3-x)^4(12-4x-15)+c\\ &= (3-x)^4(-4x-3)+c\\ &= -(3-x)^4(4x+3)+c\end{aligned}$$

8. $u=5-2x \quad du=-2\,\mathrm{d}x$
   $x=\frac{5-u}{2} \quad dx=-\frac{1}{2}\,du$

$$\begin{aligned}&\int 4x(5-2x)^5\,\mathrm{d}x\\ &= \int 4\frac{5-u}{2}u^5\left(-\frac{1}{2}\right)\,du\\ &= \int (u-5)u^5\,du\\ &= \frac{u^7}{7}-\frac{5u^6}{6}+c\\ &= \frac{1}{42}u^6(6u-35)+c\\ &= \frac{1}{42}(5-2x)^6\left(6(5-2x)-35\right)+c\\ &= \frac{1}{42}(5-2x)^6(-12x-5)+c\\ &= -\frac{1}{42}(5-2x)^6(12x+5)+c\end{aligned}$$

9. $u=2x+3 \quad du=2\,\mathrm{d}x$
   $x=\frac{u-3}{2} \quad dx=\frac{1}{2}\,du$

$$\begin{aligned}&\int 20x(2x+3)^3\,\mathrm{d}x\\ &= \int 5(u-3)u^3\,du\\ &= u^5-\frac{15u^4}{4}+c\\ &= \frac{1}{4}u^4(4u-15)+c\\ &= \frac{1}{4}(2x+3)^4\left(4(2x+3)-15\right)+c\\ &= \frac{1}{4}(2x+3)^4(8x-3)+c\end{aligned}$$

10. $u=3x+1 \quad du=3\,\mathrm{d}x$
    $x=\frac{u-1}{3} \quad dx=\frac{1}{3}\,du$

$$\begin{aligned}&\int 18x\sqrt{3x+1}\,\mathrm{d}x\\ &= \int 2(u-1)u^{\frac{1}{2}}\,du\\ &= 2\int u^{\frac{3}{2}}-u^{\frac{1}{2}}\,du\\ &= 2\left(\frac{2u^{\frac{5}{2}}}{5}-\frac{2u^{\frac{3}{2}}}{3}\right)+c\\ &= \frac{4}{15}\sqrt{u^3}(3u-5)+c\\ &= \frac{4}{15}\sqrt{(3x+1)^3}\left(3(3x+1)-5\right)+c\\ &= \frac{4}{15}\sqrt{(3x+1)^3}(9x-2)+c\end{aligned}$$

11. $u=3x^2+5 \quad du=6x\,\mathrm{d}x$

$$\begin{aligned}\int\frac{6x\,\mathrm{d}x}{\sqrt{3x^2+5}} &= \int\frac{du}{\sqrt{u}}\\ &= 2\sqrt{u}+c\\ &= 2\sqrt{3x^2+5}+c\end{aligned}$$

12. $u=1-2x \quad du=-2\,\mathrm{d}x$
    $x=\frac{1-u}{2} \quad dx=-\frac{1}{2}\,du$

$$\begin{aligned}\int \frac{3x\,\mathrm{d}x}{\sqrt{1-2x}} &= \int \frac{3(1-u)}{2\sqrt{u}}\left(-\frac{1}{2}\right)\,du\\ &= -\frac{3}{4}\int \frac{1-u}{\sqrt{u}}\,du\\ &= -\frac{3}{4}\int \frac{1}{\sqrt{u}} - \sqrt{u}\,du\\ &= -\frac{3}{4}\left(2\sqrt{u} - \frac{2}{3}u^{\frac{3}{2}}\right) + c\\ &= -\frac{1}{2}\sqrt{u}(3-u) + c\\ &= \frac{1}{2}\sqrt{u}(u-3) + c\\ &= \frac{1}{2}\sqrt{1-2x}(1-2x-3) + c\\ &= -\frac{1}{2}\sqrt{1-2x}(2x+2) + c\\ &= -\sqrt{1-2x}(x+1) + c\end{aligned}$$

13. $u = \sin 2x \quad du = 2\cos 2x\,\mathrm{d}x$

$$\begin{aligned}\int 8\sin^5 2x\cos 2x\,\mathrm{d}x &= \int 4u^5\,du\\ &= \frac{4u^6}{6} + c\\ &= \frac{2}{3}\sin^6 2x + c\end{aligned}$$

14. $u = \cos 3x \quad du = -3\sin 3x\,\mathrm{d}x$

$$\begin{aligned}\int 27\cos^7 3x\sin 3x\,\mathrm{d}x &= \int -9u^7\,du\\ &= -\frac{9}{8}u^8 + c\\ &= -\frac{9}{8}\cos^8 3x + c\end{aligned}$$

15. $u = x^2 + 4 \quad du = 2x\,\mathrm{d}x$

$$\begin{aligned}\int 6x\sin(x^2+4)\,\mathrm{d}x &= \int 3\sin u\,du\\ &= -3\cos u + c\\ &= -3\cos(x^2+4) + c\end{aligned}$$

16. $u = 2x + 1 \quad du = 2\,\mathrm{d}x$

$2x = u - 1 \quad dx = \frac{1}{2}\,du$

$$\begin{aligned}&\int (4x+3)(2x+1)^5\,\mathrm{d}x\\ &= \int (2(u-1)+3)\,u^5\frac{1}{2}\,du\\ &= \frac{1}{2}\int (2u+1)u^5\,du\\ &= \frac{1}{2}\left(\frac{2u^7}{7} + \frac{u^6}{6}\right) + c\\ &= \frac{1}{84}u^6(12u+7) + c\\ &= \frac{1}{84}(2x+1)^6\,(12(2x+1)+7) + c\\ &= \frac{1}{84}(2x+1)^6(24x+19) + c\end{aligned}$$

Exercise 7D

1. $\int (x + \sin 3x)dx = \frac{x^2}{2} - \frac{\cos 3x}{3} + c$

2. $\int 2\,\mathrm{d}x = 2x + c$

3. $\int \sin 8x\,\mathrm{d}x = -\frac{\cos 8x}{8} + c$

4. $$\begin{aligned}&\int (\cos x + \sin x)(\cos x - \sin x)dx\\ &= \int \cos^2 x - \sin^2 x\,\mathrm{d}x\\ &= \int \cos 2x\,\mathrm{d}x\\ &= \frac{1}{2}\sin 2x + c\end{aligned}$$

5. $$\begin{aligned}\int \frac{x^2 + x}{\sqrt{x}}dx &= \int x^{\frac{3}{2}} + x^{\frac{1}{2}}\,\mathrm{d}x\\ &= \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c\end{aligned}$$

6. $$\int \sin^3 2x \, dx = \int \sin 2x(1-\cos^2 2x)\, dx$$
$$= -\frac{\cos 2x}{2} + \frac{1}{2}\frac{\cos^3 2x}{3} + c$$
$$= \frac{\cos 2x}{6}(\cos^2 2x - 3) + c$$

7. $$\int (3+\cos^2 x)\, dx = \frac{1}{2}\int 6 + 2\cos^2 x \, dx$$
$$= \frac{1}{2}\int 7 + 2\cos^2 x - 1 \, dx$$
$$= \frac{1}{2}\int 7 + \cos 2x \, dx$$
$$= \frac{7}{2}x + \frac{1}{4}\sin 2x + c$$

8. $$\int 4x \sin x^2 \, dx = 2\int 2x \sin x^2 \, dx$$
$$= 2(-\cos x^2) + c$$
$$= -2\cos x^2 + c$$

9. $u = x^2 - 3 \quad du = 2x\, dx$
$$\int 8x \sin(x^2-3)\, dx = 4\int \sin u \, du$$
$$= -4\cos u + c$$
$$= -4\cos(x^2-3) + c$$

10. $u = 1+3x \quad du = 3\, dx$
$$\int 24\sqrt{1+3x}\, dx = 8\int \sqrt{u}\, du$$
$$= 8\left(\frac{2}{3}\right)u^{\frac{3}{2}} + c$$
$$= \frac{16(1+3x)^{\frac{3}{2}}}{3} + c$$

11. $u = 1+3x \quad du = 3\, dx$

$3x = u - 1 \quad dx = \frac{1}{3}du$
$$\int 15x\sqrt{1+3x}\, dx$$
$$= \int 5(u-1)\sqrt{u}\frac{1}{3}\, du$$
$$= \frac{5}{3}\int u^{\frac{3}{2}} - \sqrt{u}\, du$$
$$= \frac{5}{3}(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}) + c$$
$$= \frac{2}{3}u^{\frac{5}{2}} - \frac{10}{9}u^{\frac{3}{2}} + c$$
$$= \frac{2}{9}u^{\frac{3}{2}}(3u-5) + c$$
$$= \frac{2}{9}(1+3x)^{\frac{3}{2}}(3(1+3x)-5) + c$$
$$= \frac{2}{9}(1+3x)^{\frac{3}{2}}(3+9x-5) + c$$
$$= \frac{2}{9}(1+3x)^{\frac{3}{2}}(9x-2) + c$$

12. $$\int \sin^4 2x \cos 2x \, dx = \frac{1}{2}\int \sin^4 2x(2\cos 2x)\, dx$$
$$= \frac{1}{10}\sin^5 2x$$

13. $u = 2x+7 \quad du = 2\, dx$

$2x = u - 7 \quad dx = \frac{1}{2}du$
$$\int 6x(2x{+}7)^5\, dx$$
$$= \int 3(u-7)u^5\frac{1}{2}\, du$$
$$= \frac{3}{2}\int u^6 - 7u^5\, du$$
$$= \frac{3}{2}(\frac{1}{7}u^7 - \frac{7}{6}u^6) + c$$
$$= \frac{3}{84}u^6(6u-49) + c$$
$$= \frac{3}{84}(2x+7)^6\,(6(2x+7)-49) + c$$
$$= \frac{3}{84}(2x+7)^6(12x-7) + c$$

14. $$\int 6(2x+7)^5\, dx = 3\int 2(2x+7)^5\, dx$$
$$= \frac{1}{2}(2x+7)^6 + c$$

15. $$\int (3x^2-2)dx = x^3 - 2x + c$$

16. $$\int 4x(3x^2-3)^7\, dx = \frac{2}{3}\int 6x(3x^2-3)^7\, dx$$
$$= \frac{1}{12}(3x^2-2)^8 + c$$

17. $$\int (\cos x + \sin 2x)dx = \sin x - \frac{1}{2}\cos 2x + c$$

18. $u = 3x-2 \quad du = 3\, dx$

$3x = u + 2 \quad dx = \frac{1}{3}du$
$$\int 6x(3x{-}2)^7\, dx$$
$$= \int 2(u+2)u^7\frac{1}{3}\, du$$
$$= \frac{2}{3}\int u^8 + 2u^7\, du$$
$$= \frac{2}{3}(\frac{1}{9}u^9 + \frac{1}{4}u^8) + c$$
$$= \frac{1}{54}u^8(4u+9) + c$$
$$= \frac{1}{54}(3x-2)^8\,(4(3x-2)+9) + c$$
$$= \frac{1}{54}(3x-2)^8(12x+1) + c$$

19. $$\int x\, dx = \frac{x^2}{2} + c$$

20. $u = 1 + 2x \quad du = 2\,\mathrm{d}x$

$$\begin{aligned}\int \frac{6}{\sqrt{1+2x}}\,\mathrm{d}x &= \int \frac{3}{\sqrt{u}}\,du\\ &= 6\sqrt{u} + c\\ &= 6\sqrt{1+2x} + c\end{aligned}$$

21. $u = 1 + 2x \quad du = 2\,\mathrm{d}x$

$2x = u - 1 \quad \mathrm{d}x = \frac{1}{2}\,du$

$$\begin{aligned}\int \frac{6}{x}\sqrt{1+2x}\,\mathrm{d}x &= \int \frac{3(u-1)}{2\sqrt{u}}\,du\\ &= \frac{3}{2}\int \frac{u-1}{\sqrt{u}}\,du\\ &= \frac{3}{2}\int \sqrt{u} - \frac{1}{\sqrt{u}}\,du\\ &= \frac{3}{2}(\frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u}) + c\\ &= u^{\frac{3}{2}} - 3\sqrt{u} + c\\ &= \sqrt{u}(u-3) + c\\ &= \sqrt{1+2x}(1+2x-3) + c\\ &= \sqrt{1+2x}(2x-2) + c\end{aligned}$$

22. $$\begin{aligned}\int 6\sin 2x\cos x\,\mathrm{d}x &= \int 6(2\sin x\cos x)\cos x\,\mathrm{d}x\\ &= 12\int \sin x\cos^2 x\,\mathrm{d}x\\ &= -4\cos^3 x + c\end{aligned}$$

23. $$\begin{aligned}\int 6\cos 2x\sin x\,\mathrm{d}x &= \int 6(2\cos^2 x - 1)\sin x\,\mathrm{d}x\\ &= 6\int 2\cos^2 x\sin x - \sin x\,\mathrm{d}x\\ &= 6(-\frac{2}{3}\cos^3 x + \cos x) + c\\ &= -4\cos^3 x + 6\cos x + c\end{aligned}$$

24. $$\int (x^2+x+1)^8(2x+1)\,\mathrm{d}x = \frac{1}{9}(x^2+x+1)^9 + c$$

25. $u = x^2 + 3 \quad du = 2x\,\mathrm{d}x$

$$\begin{aligned}\int 24x\sin(x^2+3)\,\mathrm{d}x &= 12\int 2x\sin u\,\mathrm{d}x\\ &= 12\int \sin u\,du\\ &= -12\cos u + c\\ &= -12\cos(x^2+3) + c\end{aligned}$$

26. $u = x - 5 \quad du = \mathrm{d}x$

$x = u + 5 \quad \mathrm{d}x = du$

$$\begin{aligned}&\int (2x+1)\sqrt[3]{x-5}\,\mathrm{d}x\\ &= \int (2u+10+1)u^{\frac{1}{3}}\,\mathrm{d}x\\ &= \int (2u+11)u^{\frac{1}{3}}\,\mathrm{d}x\\ &= \int 2u^{\frac{4}{3}} + 11u^{\frac{1}{3}}\,\mathrm{d}x\\ &= \frac{6u^{\frac{7}{3}}}{7} + \frac{33u^{\frac{4}{3}}}{4} + c\\ &= \frac{3u^{\frac{4}{3}}}{28}(8u+77) + c\\ &= \frac{3}{28}(x-5)^{\frac{4}{3}}(8(x-5)+77) + c\\ &= \frac{3}{28}(x-5)^{\frac{4}{3}}(8x+37) + c\end{aligned}$$

27. $u = \sqrt{x} + 5 \quad du = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$

$$\begin{aligned}\int \frac{(\sqrt{x}+5)^5}{\sqrt{x}}\,\mathrm{d}x &= 2\int \frac{(\sqrt{x}+5)^5}{2\sqrt{x}}\,\mathrm{d}x\\ &= 2\int u^5\,du\\ &= \frac{2u^6}{6} + c\\ &= \frac{(\sqrt{x}+5)^6}{3} + c\end{aligned}$$

28. $$\begin{aligned}\int 4(2x-1)^5\,\mathrm{d}x &= 2\int 2(2x-1)^5\,\mathrm{d}x\\ &= \frac{2(2x-1)^6}{6} + c\\ &= \frac{(2x-1)^6}{3} + c\end{aligned}$$

29. $u = 2x - 1 \quad du = 2\,\mathrm{d}x$

$2x = u + 1$

$$\begin{aligned}&\int 4x(2x-1)^5\,\mathrm{d}x\\ &= \int 2x\,u^5\,du\\ &= \int (u+1)u^5\,du\\ &= \int u^6 + u^5\,du\\ &= \frac{u^7}{7} + \frac{u^6}{6} + c\\ &= \frac{(2x-1)^7}{7} + \frac{(2x-1)^6}{6} + c\\ &= \frac{1}{42}(2x-1)^6(6(2x-1)+7) + c\\ &= \frac{1}{42}(2x-1)^6(12x+1) + c\end{aligned}$$

30. $\int \cos^3 6x \sin 6x \,\mathrm{d}x = -\frac{1}{6}\int -6\cos^3 6x \sin 6x \,\mathrm{d}x$

$= -\frac{\cos^4 6x}{24} + c$

31. $\int \frac{6x}{\sqrt{x^2-3}}\,\mathrm{d}x = 3\int \frac{2x}{\sqrt{x^2-3}}\,\mathrm{d}x$

$= 6\sqrt{x^2-3} + c$

32. $\int \sin 2x \cos 2x \,\mathrm{d}x = \int \sin 4x \,\mathrm{d}x$

$= -\frac{\cos 4x}{4} + c$

33. $u = 2x - 1 \quad du = 2\,\mathrm{d}x$

$2x = u + 1$

$$\begin{aligned}
&\int 8x^2(2x-1)^5\,\mathrm{d}x \\
&= \int (2x)^2\,u^5\,du \\
&= \int (u+1)^2 u^5\,du \\
&= \int (u^2+2u+1)u^5\,du \\
&= \int u^7 + 2u^6 + u^5\,du \\
&= \frac{u^8}{8} + \frac{2u^7}{7} + \frac{u^6}{6} + c \\
&= \frac{u^6}{168}(21u^2 + 48u + 28) + c \\
&= \frac{(2x-1)^6}{168}(21(2x-1)^2 + 48(2x-1) + 28) + c \\
&= \frac{(2x-1)^6}{168}(21(4x^2-4x+1) + 96x - 48 + 28) + c \\
&= \frac{(2x-1)^6}{168}(84x^2 - 84x + 21 + 96x - 20) + c \\
&= \frac{(2x-1)^6}{168}(84x^2 + 12x + 1) + c
\end{aligned}$$

Exercise 7E

1. $\int \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int 6x - 5\,\mathrm{d}x$

$y = 3x^2 - 5x + c$

2. $\int \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int 6\sqrt{x}\,\mathrm{d}x$

$y = 4x^{\frac{3}{2}} + c$

3. $\int 8y\,dy = \int 4x - 1\,\mathrm{d}x$

$4y^2 = 2x^2 - x + c$

4. $\int 3y\,dy = \int \frac{5}{x^2}\,\mathrm{d}x$

$\frac{3}{2}y^2 = -\frac{5}{x} + c$

5. $6y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2}$

$\int 6y\,dy = \int \frac{dx}{x^2}$

$3y^2 = -\frac{1}{x} + c$

6. $\sin 2y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{4x^2}$

$\int \sin 2y\,dy = \int \frac{5}{4x^2}\,\mathrm{d}x$

$-\frac{\cos 2y}{2} = -\frac{5}{4x} + c$

$2\cos 2y = \frac{5}{x} + c$

7. $(2y-3)\frac{\mathrm{d}y}{\mathrm{d}x} = 8x + 1$

$\int 2y - 3\,dy = \int 8x + 1\,\mathrm{d}x$

$y^2 - 3y = 4x^2 + x + c$

8. $(4y-5)\frac{\mathrm{d}y}{\mathrm{d}x} = x(2-3x)$

$\int 4y - 5\,dy = \int 2x - 3x^2\,\mathrm{d}x$

$2y^2 - 5y = x^2 - x^3 + c$

9. $\cos y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2}$

$\int \cos y\,dy = \int \frac{1}{x^2}\,\mathrm{d}x$

$\sin y = -\frac{1}{x} + c$

10. $$2y(y^2+1)^5\frac{\mathrm{d}y}{\mathrm{d}x} = x$$
$$\int 2y(y^2+1)^5\,dy = \int x\,\mathrm{d}x$$
$$\frac{(y^2+1)^6}{6} = \frac{x^2}{2}+c$$
$$(y^2+1)^6 = 3x^2+c$$

11. $$\int dy = \int 6x\,\mathrm{d}x$$
$$y = 3x^2+c$$
$$4 = 3(-1)^2+c$$
$$c = 1$$
$$\therefore\quad y = 3x^2+1$$

12. $$6y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{x^2}$$
$$\int 6y\,dy = \int \frac{5}{x^2}\,\mathrm{d}x$$
$$3y^2 = -\frac{5}{x}+c$$
$$3(1)^2 = -\frac{5}{0.5}+c$$
$$3 = -10+c$$
$$x = 13$$
$$\therefore\quad 3y^2 = 13-\frac{5}{x}$$

13. $$\int 2+\cos y\,dy = \int 2x+3\,\mathrm{d}x$$
$$2y+\sin y = x^2+3x+c$$
$$\pi+1 = 1+3+c$$
$$c = \pi-3$$
$$\therefore\quad 2y+\sin y = x^2+3x+\pi-3$$

14. $$2y+3\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3+8x$$
$$\int 2y+3\,dy = \int 4x^3+8x\,\mathrm{d}x$$
$$y^2+3y = x^4+4x^2+c$$
$$4+6 = 1+4+c$$
$$c = 5$$
$$\therefore\quad y^2+3y = x^4+4x^2+5$$

15. $$\int v\,dv = \int 6s^2\,ds$$
$$\frac{v^2}{2} = 2s^3+c_1$$
$$v^2 = 4s^3+c$$
$$6^2 = 4(2)^3+c$$
$$36 = 32+c$$
$$c = 4$$
$$\therefore\quad v^2 = 4s^3+4$$
$$v^2 = 4(3)^3+4$$
$$= 112$$
$$v = \pm\sqrt{112}$$
$$= 4\sqrt{7}$$

(Ignore the negative solution, because v is a speed so can not be negative.)

16. $$y\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$$
$$\int y\,dy = \int -\sin x\,\mathrm{d}x$$
$$y^2 = 2\cos x+c$$
$$2^2 = 2\cos(\pi/3)+c$$
$$4 = 1+c$$
$$c = 3$$
$$\therefore\quad y^2 = 2\cos x+3$$

(a) $$a^2 = 2\cos\pi+3$$
$$a = \sqrt{-2+3}$$
$$= 1$$
$\therefore$ Point A is $(\pi, 1)$

(b) $$b^2 = 2\cos(\pi/6)+3$$
$$b = \sqrt{\sqrt{3}+3}$$
$\therefore$ Point B is $(\pi/6, \sqrt{\sqrt{3}+3})$

At B, $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sin(\pi/6)}{\sqrt{\sqrt{3}+3}}$$
$$= -\frac{1/2}{\sqrt{\sqrt{3}+3}}$$
$$= -\frac{1}{2\sqrt{\sqrt{3}+3}}$$

17. $$2V\frac{\mathrm{d}V}{\mathrm{d}t} = 25$$
$$\int 2V\,dV = \int 25\,dt$$
$$V^2 = 25t+c$$
$$20^2 = 25(0)+c$$
$$c = 400$$
$$\therefore\quad V^2 = 25t+400$$

(a) $$V^2 = 25(20)+400$$
$$= 900$$
$$V = 300\text{cm}^3$$

(b) $$40^2 = 25t+400$$
$$1600 = 25t+400$$
$$1200 = 25t$$
$$t = 48$$

Exercise 7F

1. $$\int_0^2 10x^4\,dx = \left[2x^5\right]_0^2 = 2(2^5) - 2(0^5) = 64$$

2. $$\int_2^4 2\,dx = [2x]_2^4 = 2\times 4 - 2\times 2 = 4$$

3. $$\int_1^3 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^3 = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3}$$

4. $$\int_{-1}^1 (2x-3)\,dx = \left[x^2 - 3x\right]_{-1}^1 = ((1)^2 - 3(1)) - ((-1)^2 - 3(-1)) = -2 - 4 = -6$$

5. It is not necessary to do any work for this. Any integral with equal upper and lower bounds is equal to zero.

6. $$\int_2^3 (2+6x)\,dx = \left[2x + 3x^2\right]_2^3 = (2(3) + 3(3^2)) - (2(2) + 3(2^2)) = 33 - 16 = 17$$

7. $$\int_{-1}^2 (x-1)^2\,dx = \left[\frac{1}{3}(x-1)^3\right]_{-1}^2 = \frac{1}{3}((2-1)^3 - (-1-1)^3) = \frac{1}{3}(1 - (-8)) = 3$$

8. $$\int_0^1 (2x-1)^4\,dx = \frac{1}{2}\int_0^1 2(2x-1)^4\,dx = \frac{1}{2}\left[\frac{1}{5}(2x-1)^5\right]_0^1 = \frac{1}{10}((2(1)-1)^5 - (2(0)-1)^5) = \frac{1}{10}(1-(-1)) = 0.2$$

9. $u = x+1 \quad du = dx$
   $x = u-1 \quad dx = du$

$$\int_3^8 (\frac{1}{\sqrt{x+1}}\,dx = \int_{u=4}^9 \frac{1}{\sqrt{u}}\,du = \left[2\sqrt{u}\right]_4^9 = 6 - 4 = 2$$

10. $$\int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = -(0-1) = 1$$

11. $$\int_0^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_0^{\frac{\pi}{2}} = (1-0) = 1$$

12. $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} -0.5(1 - 2\sin^2 x - 1)\,dx = -0.5(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 2x - 1\,dx = -0.5\left[\frac{\sin 2x}{2} - x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -0.5\left(\frac{\sin\frac{2\pi}{3}}{2} - \frac{\pi}{3} - \frac{\sin\frac{\pi}{3}}{2} + \frac{\pi}{6}\right) = -0.5\left(\frac{\sqrt{3}}{4} - \frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{6}\right) = -0.5\left(-\frac{\pi}{6}\right) = \frac{\pi}{12}$$

13. $$\int_{\frac{\pi}{2}}^{\pi} \cos\frac{x}{2}\,dx = \left[2\sin\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi} = 2\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right) = 2\left(1 - \frac{\sqrt{2}}{2}\right) = 2 - \sqrt{2}$$

14. $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\cos^2 x \sin x\,dx = \left[\frac{4}{3}\cos^3 x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{4}{3}\left(\cos^3\frac{\pi}{4} - \cos^3 -\frac{\pi}{4}\right) = 0$$

15. $\int 3x^2\,dx = x^3 + c$ so

   (a) $\int_0^1 3x^2\,dx = \left[x^3\right]_0^1 = 1^3 - 0^3 = 1$

   (b) $\int_1^3 3x^2\,dx = \left[x^3\right]_1^3 = 3^3 - 1^3 = 26$

   (c) $\int_0^3 3x^2\,dx = \left[x^3\right]_0^3 = 3^3 - 0^3 = 27$

16. $\int 2x + x^2 \, \mathrm{d}x = x^2 + \frac{x^3}{3} + c$ so

(a) $$\int_0^3 2x + x^2 \, \mathrm{d}x = \left[x^2 + \frac{x^3}{3}\right]_0^3 = 3^2 + \frac{3^3}{3} - 0^2 - \frac{0^3}{3} = 9 + 9 - 0 - 0 = 18$$

(b) $$\int_3^4 2x + x^2 \, \mathrm{d}x = \left[x^2 + \frac{x^3}{3}\right]_3^4 = 4^2 + \frac{4^3}{3} - 3^2 - \frac{3^3}{3} = 16 + \frac{64}{3} - 9 - 9 = \frac{58}{3}$$

(c) $$\int_0^4 2x + x^2 \, \mathrm{d}x = \left[x^2 + \frac{x^3}{3}\right]_0^4 = 4^2 + \frac{4^3}{3} - 0^2 - \frac{0^3}{3} = 16 + \frac{64}{3} - 0 - 0 = \frac{112}{3}$$

17. $\int \sin x \, \mathrm{d}x = -\cos x + c$ so

(a) $$\int_0^{\frac{\pi}{4}} \sin x \, \mathrm{d}x = [-\cos x]_0^{\frac{\pi}{4}} = -\cos\frac{\pi}{4} + \cos 0 = 1 - \frac{\sqrt{2}}{2}$$

(b) $$\int_{-\frac{\pi}{4}}^0 \sin x \, \mathrm{d}x = [\cos x]_{-\frac{\pi}{4}}^0 = -\cos 0 + \cos -\frac{\pi}{4} = \frac{\sqrt{2}}{2} - 1$$

18. (a) $$\int_0^4 x \, \mathrm{d}x = \left[\frac{x^2}{2}\right]_0^4 = \frac{4^2}{2} - \frac{0^2}{2} = 8$$

(b) $$\int_0^4 3x \, \mathrm{d}x = 3\int_0^4 x \, \mathrm{d}x = 3 \times 8 = 24$$

(c) $$\int_0^4 5x \, \mathrm{d}x = 5\int_0^4 x \, \mathrm{d}x = 5 \times 8 = 40$$

19. $u = 2x + 1 \quad du = 2dx$

$x = \frac{u-1}{2} \quad dx = \frac{du}{2}$

$$\int_0^1 16(2x+1)^3 \, \mathrm{d}x = 16\int_{u=1}^3 u^3 \, \frac{du}{2} = 16\left[\frac{u^4}{8}\right]_1^3 = 16\left(\frac{81-1}{8}\right) = 160$$

20. Using the same substitutions as the previous question,

$$\int_0^1 16x(2x+1)^3 \, \mathrm{d}x = 16\int_{u=1}^3 \left(\frac{u-1}{2}\right) u^3 \, \frac{du}{2} = 4\int_{u=1}^3 u^4 - u^3 \, du = 4\left[\frac{u^5}{5} - \frac{u^4}{4}\right]_1^3 = 4\left(\frac{243}{5} - \frac{81}{4} - \frac{1}{5} + \frac{1}{4}\right) = \frac{968}{5} - \frac{400}{5} = \frac{568}{5} = 113.6$$

21. $u = x + 5 \quad du = dx$

$x = u - 5 \quad dx = du$

$$\int_0^1 \frac{6x}{25}(x+5)^4 \, \mathrm{d}x = \int_{u=5}^6 \frac{6(u-5)}{25}(u)^4 \, du = \int_5^6 \frac{6u^5}{25} - \frac{6u^4}{5} \, du = \left[\frac{u^6}{25} - \frac{6u^5}{25}\right]_5^6 = \frac{1}{25}\left[u^5(u-6)\right]_5^6 = \frac{1}{25}\left(6^5(6-6) - 5^5(5-6)\right) = \frac{1}{25} \times 5^5 = 125$$

22. $u = \sin x \quad du = \cos x \, \mathrm{d}x$

$$\int_0^{\frac{\pi}{2}} 12\sin^5 x \cos x \, \mathrm{d}x = \int_{u=0}^1 12u^5 \, du = \left[2u^6\right]_0^1 = 2(1-0) = 2$$

23. $u = 5x + 6 \quad du = 5dx$

$x = \frac{u-6}{5} \quad dx = \frac{du}{5}$

$$\int_2^6 \frac{3x}{\sqrt{5x+6}}\,\mathrm{d}x = \int_{u=16}^{36} \frac{\frac{3(u-6)}{5}}{\sqrt{u}}\frac{du}{5}$$
$$= \int_{16}^{36} \frac{3u-18}{25\sqrt{u}}\,du$$
$$= \int_{16}^{36} \frac{3\sqrt{u}}{25} - \frac{18}{25\sqrt{u}}\,du$$
$$= \left[\frac{3u^{\frac{3}{2}}}{25}\times\frac{2}{3} - \frac{18u^{\frac{1}{2}}}{25}\times\frac{2}{1}\right]_{16}^{36}$$
$$= \left[\frac{2u^{\frac{3}{2}}}{25} - \frac{36u^{\frac{1}{2}}}{25}\right]_{16}^{36}$$
$$= \frac{2}{25}\left[\sqrt{u}(u-18)\right]_{16}^{36}$$
$$= \frac{2}{25}\left(6(36-18) - 4(16-18)\right)$$
$$= \frac{2}{25}(108+8)$$
$$= \frac{232}{25}$$
$$= 9.28$$

24. $u = x - 1 \quad du = dx$

$x = u + 1 \quad dx = du$

$$\int_2^5 \frac{x+3}{\sqrt{x-1}}\,\mathrm{d}x = \int_{u=1}^{4} \frac{u+4}{\sqrt{u}}\,du$$
$$= \int_1^4 \sqrt{u} + \frac{4}{\sqrt{u}}\,du$$
$$= \left[u^{\frac{3}{2}}\times\frac{2}{3} + 4u^{\frac{1}{2}}\times\frac{2}{1}\right]_1^4$$
$$= \left[\frac{2}{3}u^{\frac{3}{2}} + 8u^{\frac{1}{2}}\right]_1^4$$
$$= \frac{2}{3}\left[u^{\frac{3}{2}} + 12u^{\frac{1}{2}}\right]_1^4$$
$$= \frac{2}{3}\left[\sqrt{u}(u+12)\right]_1^4$$
$$= \frac{2}{3}\left(2(4+12) - 1(1+12)\right)$$
$$= \frac{2}{3}(32-13)$$
$$= \frac{38}{3}$$
$$= 12\frac{2}{3}$$

Exercise 7G

1. Area is always positive, but
$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x$$
is signed, so they will only be equal where
$$\mathrm{f}(x) \geq 0 \quad \forall x \in [a, b]$$
i.e. (a), (b), (e), and (f)

2. Since $3x^2 > 0$ for $x \in [1, 3]$, the area is given by
$$A = \int_1^3 3x^2\,\mathrm{d}x = \left[x^3\right]_1^3 = 27 - 1 = 26 \text{ units}^2$$

3. $y = 4 - x^2$ crosses the $x-$axis at $x = 2$, so

(a) $$A = \left|\int_0^2 4 - x^2\,\mathrm{d}x\right|$$
$$= \left|\left[4x - \frac{x^3}{3}\right]_0^2\right|$$
$$= \left|0 - 8 + \frac{8}{3}\right|$$
$$= \frac{16}{3} \qquad = 5\frac{1}{3} \text{ units}^2$$

(b) $$A = \left|\int_0^2 4 - x^2\,\mathrm{d}x\right| + \left|\int_2^3 4 - x^2\,\mathrm{d}x\right|$$
$$= \frac{16}{3} + \left|\left[4x - \frac{x^3}{3}\right]_2^3\right|$$
$$= \frac{16}{3} + \left|8 - \frac{8}{3} - 12 + 9\right|$$
$$= \frac{16}{3} + \frac{7}{3}$$
$$= \frac{23}{3} \qquad = 7\frac{2}{3} \text{ units}^2$$

4. (a) Area$=\left|[x^4]_0^2\right| = 16$ units^2

(b) Area$=\left|[x^4]_{-2}^0\right| + \left|[x^4]_0^2\right| = 32$ units^2

5. $y = \sin x$ is positive between $x = 0$ and $x = \pi$ so

(a) Area$=[-\cos x]_0^{\frac{\pi}{2}} = 0 + 1 = 1$ units^2

(b) Area$=[-\cos x]_0^{\pi} = 1 + 1 = 2$ units^2

6. $x-$intercepts are at $x = 0$ and $x = 1$ so we must find the area in two pieces: $x \in [-1, 0]$ and

$x \in [0, 1]$.

$$\begin{aligned}\int 4x(1-x^2)^3\,\mathrm{d}x &= -2\int -2x(1-x^2)^3\,\mathrm{d}x\\ &= -2\frac{(1-x^2)^4}{4} + c\\ &= -0.5(1-x^2)^4 + c\end{aligned}$$

$$\begin{aligned}A &= \left|\int_{-1}^{0} y\,\mathrm{d}x\right| + \left|\int_{0}^{1} y\,\mathrm{d}x\right|\\ &= \left|\left[-0.5(1-x^2)^4\right]_{-1}^{0}\right| + \left|\left[-0.5(1-x^2)^4\right]_{0}^{1}\right|\\ &= \left|-0.5(1)^4 + 0.5(0)^4\right| + \left|-0.5(0)^4 + 0.5(1)^4\right|\\ &= |-0.5| + |0.5|\\ &= 1\text{ unit}^2\end{aligned}$$

7. $x-$intercepts are at $x = \mp\sqrt{5}$.

$$\begin{aligned}A &= \int_{-\sqrt{5}}^{\sqrt{5}} 5 - x^2\,\mathrm{d}x\\ &= \left[5x - \frac{x^3}{3}\right]_{-\sqrt{5}}^{\sqrt{5}}\\ &= \left(5\sqrt{5} - \frac{(\sqrt{5})^3}{3} + 5\sqrt{5} + \frac{(-\sqrt{5})^3}{3}\right)\\ &= \left(10\sqrt{5} - \frac{5\sqrt{5}}{3} - \frac{5\sqrt{5}}{3}\right)\\ &= \sqrt{5}\left(10 - \frac{10}{3}\right)\\ &= \frac{20\sqrt{5}}{3}\text{ units}^2\end{aligned}$$

8. (a)
$$\begin{aligned}\int_0^2 \left((2x-1)^3 - 8\right)dx &= \left[\frac{(2x-1)^4}{8} - 8x\right]_0^2\\ &= \frac{3^4}{8} - 16 - \frac{(-1)^4}{8}\\ &= \frac{81}{8} - 16 - \frac{1}{8}\\ &= -6\end{aligned}$$

(b)
$$\begin{aligned}\left|\int_0^2 \left((2x-1)^3 - 8\right)dx\right| &= |-6|\\ &= 6\end{aligned}$$

(c) Roots are given by

$$\begin{aligned}(2x-1)^3 - 8 &= 0\\ (2x-1)^3 &= 8\\ 2x - 1 &= 2\\ 2x &= 3\\ x &= 1.5\end{aligned}$$

Thus the area is to be found in two parts:

$$\begin{aligned}A_1 &= \left|\int_0^{1.5} \left((2x-1)^3 - 8\right)dx\right|\\ &= \left|\left[\frac{(2x-1)^4}{8} - 8x\right]_0^{1.5}\right|\\ &= \left|\frac{2^4}{8} - 12 - \frac{1}{8}\right|\\ &= \left|2 - 12 - \frac{1}{8}\right|\\ &= 10\frac{1}{8}\\ A_2 &= \left|\int_{1.5}^{2} \left((2x-1)^3 - 8\right)dx\right|\\ &= \left|\left[\frac{(2x-1)^4}{8} - 8x\right]_{1.5}^{2}\right|\\ &= \left|\frac{3^4}{8} - 16 - \frac{2^4}{8} + 12\right|\\ &= \left|\frac{81}{8} - 16 - 2 + 12\right|\\ &= 4\frac{1}{8}\\ A &= A_1 + A_2\\ &= 14.25\text{ units}^2\end{aligned}$$

9. We know that $y = \sin^2 x$ is non-negative for all x. Referring to earlier work (e.g. 7B question 38) to integrate sin^2x we get

$$\begin{aligned}A &= \int_0^{2\pi} \sin^2 x\,\mathrm{d}x\\ &= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{2\pi}\\ &= (\pi - 0 - 0 + 0)\\ &= \pi\text{ units}^2\end{aligned}$$

10. We want $\int_a^c |\mathrm{f}(x) - \mathrm{g}(x)|\,\mathrm{d}x$, so this immediately includes both (e) and (g). Between a and b $\mathrm{f}(x) \geq \mathrm{g}(x)$ and between b and c $\mathrm{g}(x) \geq \mathrm{f}(x)$ so both parts of (b) are positive so we include that. Both (d) and (f) will give us the difference between the area from a to b and that from b to c and (c) gives us the absolute value of this difference, so these are wrong. Part (a) deals with areas between one or other curve and the $x-$ axis so this is entirely wrong.

11. $$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} |2\sin x - \sin x|\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} |\sin x|\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x \\ &= [-\cos x]_0^{\frac{\pi}{2}} \\ &= 0 + 1 \\ &= 1 \text{ unit}^2 \end{aligned}$$

12. First find where the curves intersect:

$$\begin{aligned} x^2 + 3 &= x + 5 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= -1 \\ \text{or } x &= 2 \end{aligned}$$

Decide which curve is greater between $x = -1$ and $x = 2$ by choosing any convenient point. At $x = 0$, $x + 5 > x^2 + 3$. Thus the area is given by

$$\begin{aligned} A &= \int_{-1}^{2} (x+5) - (x^2+3)\,\mathrm{d}x \\ &= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2} \\ &= (2 - \frac{8}{3} + 4) - (\frac{1}{2} + \frac{1}{3} - 2) \\ &= 6 - 2\frac{2}{3} - \frac{1}{2} - \frac{1}{3} + 2 \\ &= 4.5 \text{ units}^2 \end{aligned}$$

13. First determine the points of intersection at $x \in \{-1, 2, 5\}$.

$$\begin{aligned} A_1 &= \left|\int_{-1}^{2} \left(x^3 - 5x^2 + 6x - (x^2 + 3x - 10)\right) dx\right| \\ &= \left|\int_{-1}^{2} \left(x^3 - 6x^2 + 3x + 10\right) dx\right| \\ &= \left|\left[\frac{x^4}{4} - 2x^3 + \frac{3x^2}{2} + 10x\right]_{-1}^{2}\right| \\ &= \left|4 - 16 + 6 + 20 - (\frac{1}{4} + 2 + \frac{3}{2} - 10)\right| \\ &= |14 - (-6.25)| \\ &= 20.25 \\ A_2 &= \left|\int_{2}^{5} \left(x^3 - 5x^2 + 6x - (x^2 + 3x - 10)\right) dx\right| \\ &= \left|\left[\frac{x^4}{4} - 2x^3 + \frac{3x^2}{2} + 10x\right]_{2}^{5}\right| \\ &= |-6.25 - 14| \\ &= 20.25 \\ A &= A_1 + A_2 \\ &= 40.5 \text{ units}^2 \end{aligned}$$

14. First determine the points of intersection: $x^2 = 8$, i.e. $x = \mp 2\sqrt{2}$

$$\begin{aligned} A &= \left|\int_{-2\sqrt{2}}^{2\sqrt{2}} \left(x^2 - 8\right) dx\right| \\ &= \left|\left[\frac{x^3}{3} - 8x\right]_{-2\sqrt{2}}^{2\sqrt{2}}\right| \\ &= \left|\frac{16\sqrt{2}}{3} - 16\sqrt{2} - \left(-frac16\sqrt{2}3 + 16\sqrt{2}\right)\right| \\ &= \left|\frac{32\sqrt{2}}{3} - 32\sqrt{2}\right| \\ &= \frac{64\sqrt{2}}{3} \text{ units}^2 \end{aligned}$$

15. (a) $$\begin{aligned} A &= \int_0^{\frac{5\pi}{6}} \left(\sin x - \frac{3x}{5\pi}\right) dx \\ &= \left[-\cos x - \frac{3x^2}{10\pi}\right]_0^{\frac{5\pi}{6}} \\ &= \left(-\cos\frac{5\pi}{6} - \frac{3\left(\frac{5\pi}{6}\right)^2}{10\pi}\right) \\ &\quad - \left(-\cos 0 - \frac{3\,(0)^2}{10\pi}\right) \\ &= \left(\frac{\sqrt{3}}{2} - \frac{75\pi^2}{360\pi}\right) + 1 \\ &= \frac{\sqrt{3}}{2} - \frac{5\pi}{24} + 1 \text{ units}^2 \end{aligned}$$

(b) It should be clear from your graph that the line and curve enclose only two regions. From the symmetry of the curve and line it should also be clear that these two regions have equal area. Thus the total area enclosed is

$$\begin{aligned} A &= 2\left(\frac{\sqrt{3}}{2} - \frac{5\pi}{24} + 1\right) \\ &= \sqrt{3} - \frac{5\pi}{12} + 2 \text{ units}^2 \end{aligned}$$

16. (a) By the null factor theorem points A, B and C must be where either $\cos x = 0$ or $\sin x = 0$, i.e. A= $(\frac{\pi}{2}, 0)$, B= $(\pi, 0)$ and C= $(\frac{3\pi}{2}, 0)$.

(b) $\int 6\cos x \sin^2 x\,\mathrm{d}x = 2\sin^3 x + c$

$$\begin{aligned}
\therefore \quad A_1 &= \int_0^{\frac{\pi}{2}} 6\cos x \sin^2 x \, dx \\
&= \left[2\sin^3 x\right]_0^{\frac{\pi}{2}} \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos x \sin^2 x \, dx \\
&= -\left[2\sin^3 x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
&= -(-2 - 2) \\
&= 4
\end{aligned}$$

$\therefore$ total $A = A_1 + A_2 = 6$ units^2

17. $$\begin{aligned}
a &= 0^3 - 3(0)^2 + 3(0) + 0.8 \\
&= 0.8\text{m} \\
b &= 2.2^3 - 3(2.2)^2 + 3(2.2) + 0.8 \\
&= 3.528\text{m} \\
A &= \int_0^{2.2} x^3 - 3x^2 + 3x + 0.8 \, dx \\
&= \left[0.25x^4 - x^3 + 1.5x^2 + 0.8x\right]_0^{2.2} \\
&= (0.25(2.2)^4 - (2.2)^3 + 1.5(2.2)^2 + 0.8(2.2)) \\
&\quad - (0.25(0)^4 - (0)^3 + 1.5(0)^2 + 0.8(0)) \\
&= 4.2284\text{m}^2
\end{aligned}$$

18. (a) First consider the point $(5, 4)$ from the persepctive of the curved edge:

$$\begin{aligned}
y &= ax^2 \\
4 &= a(5)^2 \\
a &= \frac{4}{25} \\
&= 0.16
\end{aligned}$$

The gradient at $(5, 4)$ is

$$\begin{aligned}
m &= \frac{dy}{dx} \\
&= 2ax \\
&= 2 \times 0.16 \times 5 \\
&= 1.6
\end{aligned}$$

The equation of the line is

$$\begin{aligned}
y &= 1.6x + c \\
4 &= 1.6(5) + c \\
4 &= 8 + c \\
c &= -4
\end{aligned}$$

Now d:

$$\begin{aligned}
d &= 1.6(10) - 4 \\
&= 12\text{m}
\end{aligned}$$

(b) The area can be found by considering the curved section and straight section separately:

$$\begin{aligned}
A_1 &= \int_0^5 0.16x^2 \, dx \\
&= \left[\frac{0.16x^3}{3}\right]_0^5 \\
&= \frac{20}{3} \\
A_2 &= \int_5^{10} 1.6x - 4 \, dx \\
&= \left[0.8x^2 - 4x\right]_5^{10} \\
&= 80 - 40 - (20 - 20) \\
&= 40 \\
A &= A_1 + A_2 \\
&= 46\frac{2}{3}\text{m}^2
\end{aligned}$$

19. First find the points of intersection between the curve and the line.

$$\begin{aligned}
\frac{60x - x^2}{25} &= 20 \\
60x - x^2 &= 500 \\
x^2 - 60x + 500 &= 0 \\
(x - 10)(x - 50) &= 0 \\
x &= 10 \\
\text{or } x &= 50
\end{aligned}$$

Now find the $x-$values where the curve meets the water level.

$$\begin{aligned}
\frac{60x - x^2}{25} &= 0 \\
x(60 - x) &= 0 \\
x &= 0 \\
\text{or } x &= 60
\end{aligned}$$

(which values could have been deduced from the symmetry of the problem).

Now find the area in three parts:

$$\begin{aligned}
A_1 &= \int_0^{10} 20 - \frac{60x - x^2}{25}\,\mathrm{d}x \\
&= \left[20x - \frac{60x^2}{50} + \frac{x^3}{75}\right]_0^{10} \\
&= 200 - 120 + 13.33 \\
&= 93.33 \\
A_2 &= \int_{10}^{50} \frac{60x - x^2}{25} - 20\,\mathrm{d}x \\
&= \left[\frac{6x^2}{5} - \frac{x^3}{75} - 20x\right]_{10}^{50} \\
&= 3000 - 1666.67 - 1000 - (-93.33) \\
&= 426.67 \\
A_3 &= A_1 \text{ by symmetry} \\
&= 93.33 \\
A_{\text{total}} &= A_1 + A_2 + A_3 \\
&= 613\text{m}^2 \text{ to the nearest square metre}
\end{aligned}$$

20. The coordinates of point B in the cross-section as shown are $(30, 40)$. This gives the coordinate of the curve as

$$\begin{aligned}
y &= ax^2 \\
40 &= a(30)^2 \\
a &= \frac{2}{45} \\
\therefore \quad y &= \frac{2x^2}{45}
\end{aligned}$$

From this the area of region ABC is

$$\begin{aligned}
A &= \int_0^{30} 40 - \frac{2x^2}{45}\,\mathrm{d}x \\
&= \left[40x \frac{2x^3}{135}\right]_0^{30} \\
&= 800\text{cm}^2
\end{aligned}$$

Hence the total area of the cross section is 1600cm^2 and the capacity is

$$\begin{aligned}
V &= 1600 \times 200 \\
&= 320\,000\text{cm}^3 \\
&= 320\text{L or } 0.32\text{m}^3
\end{aligned}$$

21. $$\begin{aligned}
A &= \int_0^{60} \frac{600 + 60x - x^2}{30} - \frac{4500 + 60x - x^2}{225} dx \\
&= \int_0^{60} 20 + 2x - \frac{x^2}{30} - \left(20 + \frac{4x}{15} - \frac{x^2}{225}\right) dx \\
&= \int_0^{60} \frac{26x}{15} - \frac{13x^2}{450}\,\mathrm{d}x \\
&= \left[\frac{13x^2}{15} - \frac{13x^3}{1350}\right]_0^{60} \\
&= 1040\text{m}^2
\end{aligned}$$

22. The first two $x-$intercepts can be easily determined to be where $x = 0$ and $x = 1$. For the third,

$$\begin{aligned}
\sin\left(\frac{4\pi}{3}(x-1)\right) &= 0 \\
\frac{4\pi}{3}(x-1) &= \pi \\
x - 1 &= \frac{3}{4} \\
x &= 1.75
\end{aligned}$$

Thus the area is

$$\begin{aligned}
A_1 &= \left|\int_0^1 -\sin(\pi x)\,\mathrm{d}x\right| \\
&= \left|\left[\frac{\cos(\pi x)}{\pi}\right]_0^1\right| \\
&= \left|\frac{\cos\pi - \cos 0}{\pi}\right| \\
&= \frac{2}{\pi} \\
A_2 &= \left|\int_1^{1.75} -\frac{1}{2}\sin\left(\frac{4\pi}{3}(x-1)\right)\,\mathrm{d}x\right| \\
&= \left|\left[\frac{3\cos\left(\frac{4\pi}{3}(x-1)\right)}{8\pi}\right]_1^{1.75}\right| \\
&= \frac{3}{8\pi}\left|\frac{\cos\pi - \cos 0}{\pi}\right| \\
&= \frac{3}{4\pi} \\
A_{\text{total}} &= A_1 + A_2 \\
&= \frac{11}{4\pi} \text{ units}^2
\end{aligned}$$

Miscellaneous Exercise 7

1. $$\begin{aligned}\frac{dy}{dx} &= 3(2x+1)^2(2)\\ &= 6(2x+1)^2\end{aligned}$$

2. $$\frac{dy}{dx} = -12\sin 3x + 12\cos 4x$$

3. $$\begin{aligned}\frac{dy}{dx} &= \frac{(4\sin^3 x\cos x)x - \sin^4 x}{x^2}\\ &= \frac{\sin^3 x(4x\cos x - \sin x}{x^2}\end{aligned}$$

4. $$\begin{aligned}\frac{dy}{dx} &= \frac{2\cos x(1+\cos x) - (1+2\sin x)(-\sin x)}{(1+\cos x)^2}\\ &= \frac{2\cos x + 2\cos^2 x + \sin x + 2\sin^2 x}{(1+\cos x)^2}\\ &= \frac{2\cos x + \sin x + 2(\cos^2 x + \sin^2 x)}{(1+\cos x)^2}\\ &= \frac{2\cos x + \sin x + 2}{(1+\cos x)^2}\end{aligned}$$

5. $$\begin{aligned}\frac{dy}{dx} &= \frac{2\cos 2x(1+\sin 2x) - \sin 2x(2\cos 2x)}{(1+\sin 2x)^2}\\ &= \frac{2\cos 2x + 2\sin 2x\cos 2x - 2\sin 2x\cos 2x}{(1+\sin 2x)^2}\\ &= \frac{2\cos 2x}{(1+\sin 2x)^2}\end{aligned}$$

6. $$\begin{aligned}5y + 5x\frac{dy}{dx} + 6y^2\frac{dy}{dx} &= 6x\\ 5x\frac{dy}{dx} + 6y^2\frac{dy}{dx} &= 6x - 5y\\ (5x+6y^2)\frac{dy}{dx} &= 6x - 5y\\ \frac{dy}{dx} &= \frac{6x-5y}{5x+6y^2}\end{aligned}$$

7. $$\begin{aligned}\frac{dx}{dt} &= 6t - 5\\ \frac{dy}{dt} &= -12t^2\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{-12t^2}{6t-5}\end{aligned}$$

8. $$\begin{aligned}\cos y + x(-\sin y\frac{dy}{dx}) &= \frac{dy}{dx}\sin x + y\cos x\\ \cos y - y\cos x &= \frac{dy}{dx}(\sin x + x\sin y)\\ \frac{dy}{dx} &= \frac{\cos y - ycosx}{\sin x + x\sin y}\end{aligned}$$

9. Given $\sin A = \frac{3}{5}$, we can first deduce $\cos A = \pm\frac{4}{5}$ (depending on whether A is in the first or second quadrant).

   (a) $$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ &= 2\left(\frac{3}{5}\right)\left(\pm\frac{4}{5}\right)\\ &= \pm\frac{24}{25}\end{aligned}$$

   (b) $$\begin{aligned}\cos 2A &= 1 - 2\sin^2 A\\ &= 1 - 2\left(\frac{3}{5}\right)^2\\ &= \pm\frac{7}{25}\end{aligned}$$

   (c) $$\begin{aligned}\tan 2A &= \frac{\sin 2A}{\cos 2A}\\ &= \pm\frac{24}{7}\end{aligned}$$

10. (a) $$\begin{aligned}2x + y + x\frac{dy}{dx} &= 2y\frac{dy}{dx}\\ 2x + y &= (2y - x)\frac{dy}{dx}\\ \frac{dy}{dx} &= \frac{2x+y}{2y-x}\\ &= \frac{4+3}{6-2}\\ &= \frac{7}{4}\\ (y - y_1) &= m(x - x_1)\\ y - 3 &= \frac{7}{4}(x-2)\\ 4y - 12 &= 7x - 14\\ 7x - 4y &= 2\end{aligned}$$

    (b) $$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= -\frac{x^2}{y^2}\\ &= -\frac{4}{9}\\ (y - y_1) &= m(x - x_1)\\ y - 3 &= -\frac{4}{9}(x-2)\\ 9(y-3) &= -4(x-2)\\ 9y - 27 &= -4x + 8\\ 4x + 9y &= 35\end{aligned}$$

11. (a) $$\frac{4\sin 8x}{8} + c = 0.5\sin 8x + c$$

    (b) $$\frac{(3+x^2)^6}{6} + c$$

    (c) Let $u = x + 3 \quad du = dx$
    $x = u - 3 \quad dx = du$

$$\int (2-3x)\sqrt[3]{x+3}\,\mathrm{d}x$$
$$= \int (2-3(u-3))\sqrt[3]{u}\,du$$
$$= \int (2-3u+9)u^{\frac{1}{3}}\,du$$
$$= \int 11u^{\frac{1}{3}} - 3u^{\frac{4}{3}}\,du$$
$$= 11u^{\frac{4}{3}} \times \frac{3}{4} - 3u^{\frac{7}{3}} \times \frac{3}{7} + c$$
$$= \frac{33}{4}u^{\frac{4}{3}} - \frac{9}{7}u^{\frac{7}{3}} + c$$
$$= \frac{3u^{\frac{4}{3}}}{28}(77-12u) + c$$
$$= \frac{3(x+3)^{\frac{4}{3}}}{28}(77-12(x+3)) + c$$
$$= \frac{3(x+3)^{\frac{4}{3}}}{28}(41-12x) + c$$

(d) $\dfrac{\sin^6 2x}{6} \times \dfrac{1}{2} + c = \dfrac{\sin^6 2x}{1}2 + c$

(e)
$$\int \sin^2 \frac{x}{2}\,\mathrm{d}x = -0.5\int -2\sin^2\frac{x}{2}\,\mathrm{d}x$$
$$= -0.5\int \left(1-2\sin^2\frac{x}{2}\right) - 1\,\mathrm{d}x$$
$$= -0.5\int \cos x - 1\,\mathrm{d}x$$
$$= -0.5(\sin x - x) + c$$
$$= \frac{x}{2} - \frac{\sin x}{2} + c$$

(f)
$$\int \cos^3\frac{x}{2}\,\mathrm{d}x = \int \cos\frac{x}{2}\cos^2\frac{x}{2}\,\mathrm{d}x$$
$$= \int \cos\frac{x}{2}\left(1-\sin^2\frac{x}{2}\right)\,\mathrm{d}x$$
$$= \int \cos\frac{x}{2} - \cos\frac{x}{2}\sin^2\frac{x}{2}\,\mathrm{d}x$$
$$= 2\sin\frac{x}{2} - 2\left(\frac{1}{3}\sin^3\frac{x}{2}\right) + c$$
$$= 2\sin\frac{x}{2} - \frac{2}{3}\sin^3\frac{x}{2} + c$$

12. (a) Change the sign of the argument:

$$\bar{z} = 8\operatorname{cis} -\frac{\pi}{6}$$

(b) A complex number plus its conjugate gives double the real component:

$$2 \times 8\cos\frac{\pi}{6} = 8\sqrt{3} = 8\sqrt{3}\operatorname{cis} 0$$

(c) A complex number minus its conjugate gives double the imaginary part:

$$2 \times 8\mathrm{i}\sin\frac{\pi}{6} = 8\mathrm{i} = 8\operatorname{cis}\frac{\pi}{2}$$

(d) When multiplying complex numbers in polar form, add the arguments and multiply the moduli:

$$8^2\operatorname{cis}\left(\frac{\pi}{6} + -\frac{\pi}{6}\right) = 64\operatorname{cis} 0$$

(e) When dividing complex numbers in polar form, subtract the arguments and divide the moduli:

$$\frac{8}{8}\operatorname{cis}\left(\frac{\pi}{6} - -\frac{\pi}{6}\right) = 1\operatorname{cis}\frac{\pi}{3}$$

13. L.H.S.:

$$\frac{\sin\theta}{1-\cos\theta} + \frac{1-\cos\theta}{\sin\theta}$$
$$= \frac{\sin^2\theta + (1-\cos\theta)^2}{\sin\theta(1-\cos\theta)}$$
$$= \frac{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}{\sin\theta(1-\cos\theta)}$$
$$= \frac{2-2\cos\theta}{\sin\theta(1-\cos\theta)}$$
$$= \frac{2(1-\cos\theta)}{\sin\theta(1-\cos\theta)}$$
$$= \frac{2}{\sin\theta}$$
$$= \text{R.H.S.}$$

□

14.
$$\text{L.H.S.} = \frac{\mathrm{d}}{\mathrm{d}x}(3x^2 - x + 1)$$
$$= \lim_{h\to 0}\frac{\left((3(x+h)^2 - (x+h) + 1\right) - (3x^2 - x + 1)}{h}$$
$$= \lim_{h\to 0}\frac{3x^2 + 6xh + 3h^2 - x - h + 1 - 3x^2 + x - 1}{h}$$
$$= \lim_{h\to 0}\frac{6xh + 3h^2 - h}{h}$$
$$= \lim_{h\to 0}(6x + 3h - 1)$$
$$= 6x - 1$$
$$= \text{R.H.S.}$$

□

15. To prove: for z a non-zero complex number,

$$\frac{1}{z} = \frac{\bar{z}}{|z|}$$

Proof:
Let $z = a + b\mathrm{i}$ for $a, b \in \Re$. Then $\bar{z} = a - b\mathrm{i}$ and $|z| = a^2 + b^2$.

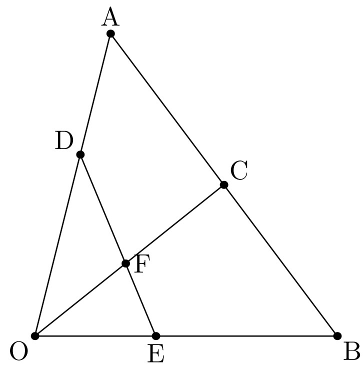
$$\text{L.H.S.} = \frac{1}{z}$$
$$= \frac{1}{a+b\mathrm{i}}$$
$$= \frac{1}{a+b\mathrm{i}} \times \frac{a-b\mathrm{i}}{a-b\mathrm{i}}$$
$$= \frac{a-b\mathrm{i}}{a^2 - b^2\mathrm{i}^2}$$
$$= \frac{a-b\mathrm{i}}{a^2+b^2}$$
$$= \frac{\bar{z}}{|z|}$$
$$= \text{R.H.S.}$$

□

16. $y = \cos^3 x$ is non-negative between $x = 0$ and $x = \frac{\pi}{2}$ so the area is the simple integral

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \cos^3 x \,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \cos x (cos^2 x) \,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \cos x (1 - \sin^2 x) \,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \cos x - \cos x \sin^2 x \,\mathrm{d}x \\ &= \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{2}} \\ &= \left(1 - \frac{1}{3}\right) - (0 - 0) \\ &= \frac{2}{3} \text{ units}^2 \end{aligned}$$

17.



$$\begin{aligned} \overrightarrow{\mathrm{AB}} &= \overrightarrow{\mathrm{AO}} + \overrightarrow{\mathrm{OB}} \\ &= \mathbf{b} - \mathbf{a} \\ \overrightarrow{\mathrm{AC}} &= 0.5\overrightarrow{\mathrm{AB}} \\ &= 0.5(\mathbf{b} - \mathbf{a}) \\ \overrightarrow{\mathrm{OC}} &= \overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{AC}} \\ &= \mathbf{a} + 0.5(\mathbf{b} - \mathbf{a}) \\ &= 0.5(\mathbf{a} + \mathbf{b}) \end{aligned}$$

(a) $$\begin{aligned} \overrightarrow{\mathrm{DF}} &= \overrightarrow{\mathrm{DO}} + \overrightarrow{\mathrm{OF}} \\ &= -h\mathbf{a} + m\overrightarrow{\mathrm{OC}} \\ &= -h\mathbf{a} + 0.5m(\mathbf{a} + \mathbf{b}) \\ &= (0.5m - h)\mathbf{a} + 0.5m\mathbf{b} \end{aligned}$$

(b) $$\begin{aligned} \overrightarrow{\mathrm{FE}} &= \overrightarrow{\mathrm{FO}} + \overrightarrow{\mathrm{OE}} \\ &= -m\overrightarrow{\mathrm{OC}} + k\mathbf{b} \\ &= -0.5m(\mathbf{a} + \mathbf{b}) + k\mathbf{b} \\ &= -0.5m\mathbf{a} + (k - 0.5m)\mathbf{b} \end{aligned}$$

(c) $$\begin{aligned} \overrightarrow{\mathrm{DF}} &= \overrightarrow{\mathrm{FE}} \\ (0.5m - h)\mathbf{a} + 0.5m\mathbf{b} &= -0.5m\mathbf{a} + (k - 0.5m)\mathbf{b} \\ (m - h)\mathbf{a} &= (k - m)\mathbf{b} \\ \therefore \quad m - h &= 0 \\ m &= h \\ \text{and} \quad k - m &= 0 \\ k &= m \\ \therefore \quad h = k &= m \end{aligned}$$

□

(d) $$\begin{aligned} \overrightarrow{\mathrm{DE}} &= \overrightarrow{\mathrm{DO}} + \overrightarrow{\mathrm{OE}} \\ &= -h\mathbf{a} + k\mathbf{b} \\ &= h(\mathbf{b} - \mathbf{a}) \\ &= h\overrightarrow{\mathrm{AB}} \\ \therefore \quad \overrightarrow{\mathrm{DE}} &\parallel \overrightarrow{\mathrm{AB}} \end{aligned}$$

□

18. (a) $$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= 2t \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= 2 - \frac{1}{t^2} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{2t}{2 - \frac{1}{t^2}} \\ &= \frac{2t^3}{2t^2 - 1} \end{aligned}$$

(b) $$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{6t^2(2t^2 - 1) - 2t^3(4t)}{(2t^2 - 1)^2} \times \frac{1}{2 - \frac{1}{t^2}} \\ &= \frac{12t^4 - 6t^2 - 8t^4}{(2t^2 - 1)^2} \times \frac{t^2}{2t^2 - 1} \\ &= \frac{(4t^4 - 6t^2)(t^2)}{(2t^2 - 1)^3} \\ &= \frac{2t^4(2t^2 - 3)}{(2t^2 - 1)^3} \end{aligned}$$

19. (a) For the first statement:

$$\begin{aligned} \text{L.H.S.} &= \cos(A - B) + \cos(A + B) \\ &= (\cos A \cos B + \sin A \sin B) \\ &\quad + (\cos A \cos B - \sin A \sin B) \\ &= 2\cos A \cos B \\ &= \text{R.H.S.} \end{aligned}$$

□

Similarly the second statement:

$$\begin{aligned} \text{L.H.S.} &= \cos(A - B) - \cos(A + B) \\ &= (\cos A \cos B + \sin A \sin B) \\ &\quad - (\cos A \cos B - \sin A \sin B) \\ &= 2\sin A \sin B \\ &= \text{R.H.S.} \end{aligned}$$

□

(b) From the preceding we can draw the following simplifications:

$$\begin{aligned} 2\sin 4x \sin x &= \cos(4x - x) - \cos(4x + x) \\ &= \cos 3x - \cos 5x \end{aligned}$$

and

$$\begin{aligned} 6\cos 7x \cos 2x &= 3\left(\cos(7x - 2x) \right. \\ &\quad \left. + \cos(7x + 2x)\right) \\ &= 3(\cos 5x + \cos 9x) \end{aligned}$$

From this,

i. $\int (2\sin 4x \sin x)dx$

$$= \int (\cos 3x - \cos 5x)dx$$
$$= \frac{1}{3}\sin 3x - \frac{1}{5}\sin 5x + c$$

ii. $\int (6\cos 7x \cos 2x)dx$

$$= 3\int (\cos 5x + \cos 9x)dx$$
$$= \frac{3}{5}\sin 5x + \frac{1}{3}\sin 9x + c$$

20. (a) $\int_0^{\frac{5\pi}{6}} (0.5 - \sin x)\,dx$

$$= [0.5x + \cos x]_0^{\frac{5\pi}{6}}$$
$$= \left(\frac{5\pi}{12} + \cos\frac{5\pi}{6}\right) - (0 + \cos 0)$$
$$= \frac{5\pi}{12} - \frac{\sqrt{3}}{2} - 1$$
$$= \frac{5\pi - 6\sqrt{3} - 12}{12}$$

(b) $\int_0^{\frac{5\pi}{6}} (0.5 - \sin x)\,dx$

$$= \frac{|5\pi - 6\sqrt{3} - 12|}{12}$$
$$= \frac{6\sqrt{3} + 12 - 5\pi}{12}$$

(If you had to do this without a calculator, you should be able to estimate the value of 5π (about 15 or 16) and $6\sqrt{3}$ (more than 6 and less than 12) sufficiently to be able to be confident that $5\pi - 6\sqrt{3} - 12 < 0$.)

(c) $0.5 - \sin x$ crosses the $x-$axis where $\sin x = 0.5$, i.e. at $x = \frac{\pi}{6}$. We need to find the area in two parts:

$$A_1 = \int_0^{\frac{\pi}{6}} (0.5 - \sin x)\,dx$$
$$= [0.5x + \cos x]_0^{\frac{\pi}{6}}$$
$$= \left(\frac{\pi}{12} + \cos\frac{\pi}{6}\right) - (0 + \cos 0)$$
$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$
$$= \frac{\pi + 6\sqrt{3} - 12}{12}$$

$$A_2 = -\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (0.5 - \sin x)\,dx$$
$$= -[0.5x + \cos x]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$
$$= -\left(\frac{5\pi}{12} + \cos\frac{5\pi}{6}\right) - \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)$$
$$= -\left(\frac{5\pi}{12} - \frac{\sqrt{3}}{2} - \frac{\pi}{12} - \frac{\sqrt{3}}{2}\right)$$
$$= \frac{12\sqrt{3} - 4\pi}{12}$$

$$A_{\text{total}} = A_1 + A_2$$
$$= \frac{\pi + 6\sqrt{3} - 12}{12} + \frac{12\sqrt{3} - 4\pi}{12}$$
$$= \frac{18\sqrt{3} - 3\pi - 12}{12}$$
$$= \frac{6\sqrt{3} - \pi - 4}{4} \text{ units}^2$$

21. John's conjecture is easily disproven by finding a single counter-example (e.g. $x = 6$ or $x = 10$). However even these give a result that is a multiple of six.

Conjecture: $x^3 - x$ is a multiple of 6 for $x \geq 2$.

Proof: Consider the factorisation

$$x^3 - x = x(x-1)(x+1)$$

At least one of these factors must be even. (If x is not even, then both $x - 1$ and $x + 1$ are even.)

Similarly, and one factor must be a multiple of 3. (If x has a remainder of 1 when divided by 3, then $x-1$ is a multiple of 3. If x has a remainder of 2 when divided by 3, then $x + 1$ is a multiple of 3. If x has no remainder when divided by 3, it is itself a multiple of 3. There are no other possibilities.)

Since $x^3 - x$ is has both a factor that is even and a factor that is a multiple of 3, it must be a multiple of 6. □

22. Let A be the given position of the shuttle, O be the position of the satellite and P be the position of the shuttle at closest approach t seconds later.

$$\overrightarrow{OP} \cdot \overrightarrow{AP} = 0$$
$$\left(\overrightarrow{OA} + \overrightarrow{AP}\right) \cdot \overrightarrow{AP} = 0$$
$$(\mathbf{r} + t\mathbf{v}) \cdot \mathbf{v} = 0$$
$$\begin{pmatrix} 30\,000 \\ -39\,000 \\ 12\,750 \end{pmatrix} \cdot \begin{pmatrix} -150 \\ 180 \\ -60 \end{pmatrix} + t\begin{pmatrix} -150 \\ 180 \\ -60 \end{pmatrix} \cdot \begin{pmatrix} -150 \\ 180 \\ -60 \end{pmatrix} = 0$$
$$-12\,285\,000 + 58\,500t = 0$$
$$t = 210\text{s}$$

$$|\overrightarrow{OP}| = |\mathbf{r} + 210\mathbf{v}|$$
$$= 1930\text{m (to the nearest 10m)}$$

Let B be the position of the shuttle at $t = 100$. Let $\mathbf{w}$ be the new velocity.

$$\overrightarrow{OB} = \mathbf{r} + 100\mathbf{v}$$
$$= \begin{pmatrix} 15\,000 \\ -21000 \\ 6\,750 \end{pmatrix} \text{m}$$
$$\mathbf{w} = -\frac{1}{12.5 \times 60}\begin{pmatrix} 15\,000 \\ -21000 \\ 6\,750 \end{pmatrix}$$
$$= \begin{pmatrix} -20 \\ 28 \\ -9 \end{pmatrix} \text{ms}^{-1}$$

23. Starting with the first expression stripped of the $+c$:

$$\frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$$
$$= \frac{3x}{8} + \frac{2\sin x\cos x}{4} + \frac{2\sin 2x\cos 2x}{32}$$
$$= \frac{3x}{8} + \frac{\sin x\cos x}{2} + \frac{\sin 2x\cos 2x}{16}$$
$$= \frac{3x}{8} + \frac{\sin x\cos x}{2} + \frac{(2\sin x\cos x)(2\cos^2 x - 1)}{16}$$
$$= \frac{3x}{8} + \frac{\sin x\cos x}{2} + \frac{4\sin x\cos^3 x - 2\sin x\cos x}{16}$$
$$= \frac{3x}{8} + \frac{\sin x\cos x}{2} + \frac{\sin x\cos^3 x}{4} - \frac{\sin x\cos x}{8}$$
$$= \frac{\sin x\cos^3 x}{4} + \frac{3\sin x\cos x}{8} + \frac{3x}{8}$$