Chapter 3

Exercise 3A

1. No working needed.

2. No working needed.

3. (a) A + B cannot be determined because the matrices are not the same size.

   (b) $A + C = \begin{bmatrix} 1+2 & 2-3 \\ 0+1 & -4-5 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & -9 \end{bmatrix}$

   (c) $C - A = \begin{bmatrix} 2-1 & -3-2 \\ 1-0 & -5+4 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$

   (d) $2D = \begin{bmatrix} 2\times 3 \\ 2\times 1 \\ 2\times -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix}$

   (e) $3B = \begin{bmatrix} 3\times 3 & 3\times -1 \\ 3\times 2 & 3\times 4 \\ 3\times 0 & 3\times 3 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ 6 & 12 \\ 0 & 9 \end{bmatrix}$

   (f) B + D cannot be determined because the matrices are not the same size.

   (g) $2A = \begin{bmatrix} 2\times 1 & 2\times 2 \\ 2\times 0 & 2\times -4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -8 \end{bmatrix}$

   (h) $2A - C = \begin{bmatrix} 2-2 & 4+3 \\ 0-1 & -8+5 \end{bmatrix}$
   $= \begin{bmatrix} 0 & 7 \\ -1 & -3 \end{bmatrix}$

4. (a) $P + Q = \begin{bmatrix} 3+2 & 2+1 & -1+0 \\ 1+0 & 4-1 & 3+0 \end{bmatrix}$
   $= \begin{bmatrix} 5 & 3 & -1 \\ 1 & 3 & 3 \end{bmatrix}$

   (b) $Q - P = \begin{bmatrix} 2-3 & 1-2 & 0+1 \\ 0-1 & -1-4 & 0-3 \end{bmatrix}$
   $= \begin{bmatrix} -1 & -1 & 1 \\ -1 & -5 & -3 \end{bmatrix}$

   (c) $3R = \begin{bmatrix} 3\times 1 & 3\times 2 & 3\times 1 \\ 3\times 2 & & \\ 3\times 1 & 3\times 2 & \end{bmatrix}$
   $= \begin{bmatrix} 3 & 6 & 3 \\ 6 & 3 & 6 \end{bmatrix}$

   (d) $3P - 2Q$
   $= \begin{bmatrix} 3\times 3-2\times 2 & 3\times 2-2\times 1 & 3\times -1-2\times 0 \\ 3\times 1-2\times 0 & 3\times 4-2\times -1 & 3\times 3-2\times 0 \end{bmatrix}$
   $= \begin{bmatrix} 5 & 4 & -3 \\ 3 & 14 & 9 \end{bmatrix}$

5. (a) A + B cannot be determined because the matrices are not the same size.

   (b) $3A = \begin{bmatrix} 3\times 2 & 3\times 4 \\ 3\times 1 & 3\times 3 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 3 & 9 \end{bmatrix}$

   (c) B+2C
   $= \begin{bmatrix} 2+2\times 3 & 1+2\times 1 & 3+2\times 4 \end{bmatrix}$
   $= \begin{bmatrix} 8 & 3 & 11 \end{bmatrix}$

   (d) C + D cannot be determined because the matrices are not the same size.

6. (a) A + B cannot be determined because the matrices are not the same size.

   (b) A+C
   $= \begin{bmatrix} 1+5 & 3+1 & 0+3 & 1-1 \\ 0+2 & 1+1 & 2+4 & 3+3 \\ 0+1 & 0+5 & 1+2 & 4+0 \end{bmatrix}$
   $= \begin{bmatrix} 6 & 4 & 3 & 0 \\ 2 & 2 & 6 & 6 \\ 1 & 5 & 3 & 4 \end{bmatrix}$

   (c) $2B = \begin{bmatrix} 2\times 3 & 2\times 1 & 2\times 4 \\ 2\times 2 & 2\times 1 & 2\times -3 \\ 2\times 0 & 2\times 1 & 2\times 2 \\ 2\times 1 & 2\times 0 & 2\times 0 \end{bmatrix}$
   $= \begin{bmatrix} 6 & 2 & 8 \\ 4 & 2 & -6 \\ 0 & 2 & 4 \\ 2 & 0 & 0 \end{bmatrix}$

   (d) $5A - C$
   $= \begin{bmatrix} 5\times 1-5 & 5\times 3-1 & 5\times 0-3 & 5\times 1+1 \\ 5\times 0-2 & 5\times 1-1 & 5\times 2-4 & 5\times 3-3 \\ 5\times 0-1 & 5\times 0-5 & 5\times 1-2 & 5\times 4-0 \end{bmatrix}$
   $= \begin{bmatrix} 0 & 14 & -3 & 6 \\ -2 & 4 & 6 & 12 \\ -1 & -5 & 3 & 20 \end{bmatrix}$

7. No working required. ‘Yes’ or ‘No’ for additions or subtractions is determined by whether the matrices specified are the same size. Multiplication by a scalar can always be determined.

8. No working required.

9. No working required.

10. $3A - 2C = B$
    $2C = 3A - B$
    $C = \frac{1}{2}(3A - B)$
    $= \frac{1}{2}\begin{bmatrix} 2 & 4 & -6 \\ 2 & 0 & -4 \end{bmatrix}$
    $= \begin{bmatrix} 1 & 2 & -3 \\ 1 & 0 & -2 \end{bmatrix}$

11. (a) Add the four individual game matrices.

    (b) Multiply the result from (a) by $\frac{1}{4}$.

12. Let the two matrices provided by A and B. The

required forecast is given by

$$\begin{aligned} &1.1(\text{A}+\text{B}) \\ &= 1.1\begin{bmatrix} 5600 & 1750 & 2320 & 1770 & 4250 \\ 2840 & 1270 & 1370 & 1020 & 2720 \\ 5050 & 1470 & 2820 & 1280 & 2700 \\ 2190 & 940 & 1520 & 840 & 1780 \end{bmatrix} \\ &= \begin{bmatrix} 6160 & 1925 & 2552 & 1947 & 4675 \\ 3124 & 1397 & 1507 & 1122 & 2992 \\ 5555 & 1617 & 3102 & 1408 & 2970 \\ 2409 & 1034 & 1672 & 924 & 1958 \end{bmatrix} \end{aligned}$$

13. No working needed. (Just substitute row and column into the expression to find the value for each element.)

14. No working needed. (Just substitute row and column into the expression to find the value for each element.)

Exercise 3B

1. $$\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1\times 2+2\times 1 & 1\times 3+2\times 3 \end{bmatrix} = \begin{bmatrix} 4 & 9 \end{bmatrix}$$

2. Not possible: the number of columns in the first matrix (2) does not equal the number of rows in the second (1).

3. $$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2\times 1-1\times 0 & 2\times 4-1\times -2 \\ 1\times 1+0\times 0 & 1\times 4+0\times -2 \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ 1 & 4 \end{bmatrix}$$

4. $$\begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3\times 1+1\times 4 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}$$

5. $$\begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix} = \begin{bmatrix} 1\times 3 & 1\times 1 \\ 4\times 3 & 4\times 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix}$$

6. $$\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2\times 2-3\times -3 & 2\times 1-3\times 2 \\ -1\times 2+4\times -3 & -1\times 1+4\times 2 \end{bmatrix} = \begin{bmatrix} 13 & -4 \\ -14 & 7 \end{bmatrix}$$

7. $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1\times 2+0\times 1 & 1\times 3+0\times -1 \\ 0\times 2+1\times 1 & 0\times 3+1\times -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

8. $$\begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times 1+4\times 0 & 1\times 0+4\times 1 \\ -1\times 1+4\times 0 & -1\times 0+3\times 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}$$

9. $$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0\times 2+0\times 4 & 0\times 1+0\times 5 \\ 0\times 2+0\times 4 & 0\times 1+0\times 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

10. $$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 3\times 2+1\times -5 & 3\times -1+1\times 3 \\ 5\times 2+2\times -5 & 5\times -1+2\times 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

11. $$\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 8\times 2-5\times 3 & 8\times 5-5\times 8 \\ -3\times 2+2\times 3 & -3\times 5+2\times 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

12. $$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix} = \begin{bmatrix} 3\times 0.5+1\times -0.5 & 3\times -0.5+1\times 1.5 \\ 1\times 0.5+1\times -0.5 & 1\times -0.5+1\times 1.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

13. $\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$
$= \begin{bmatrix} 1\times 2 + 2\times 1 + 1\times 2 + 2\times 1 \end{bmatrix}$
$= \begin{bmatrix} 8 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ 2 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix}$
$= \begin{bmatrix} 1+2 & 0+2 & 1+2 \\ 3+1 & -1+4 & 0+1 \end{bmatrix}$
$= \begin{bmatrix} 3 & 2 & 3 \\ 4 & 3 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 5 \\ 5 & 1 & -1 \end{bmatrix}$
$= \begin{bmatrix} 1+0 & 0+0 & 5+0 \\ 0+10 & 0+2 & 0-2 \\ 1+5 & 0+1 & 5-1 \end{bmatrix}$
$= \begin{bmatrix} 1 & 0 & 5 \\ 10 & 2 & -2 \\ 6 & 1 & 4 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 1 \\ -3 & -2 \end{bmatrix}$
$= \begin{bmatrix} 1+12-3 & 2+3-2 \\ 3+0+6 & 6+0+4 \end{bmatrix}$
$= \begin{bmatrix} 10 & 3 \\ 9 & 10 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+4+9 \\ 4+10+18 \end{bmatrix}$
$= \begin{bmatrix} 14 \\ 32 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix}$
$= \begin{bmatrix} 2+0+0 & 2+2+0 & -2+3+0 \\ -1+0+6 & -1+6+2 & 1+9+8 \\ 0+0+12 & 0+4+4 & 0+6+16 \end{bmatrix}$
$= \begin{bmatrix} 2 & 4 & 1 \\ 5 & 7 & 18 \\ 12 & 8 & 22 \end{bmatrix}$

19. (a) $AB = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 0+0+0 & 1+0+1 & 2+0-1 \\ 0+0+0 & 2+0-1 & 4+0+1 \\ 0+2+0 & 0+1-1 & 0+0+1 \end{bmatrix}$
$= \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 0+2+0 & 0+0+2 & 0+1+2 \\ 2+2+0 & 0+0+0 & -2+1+0 \\ 0-2+0 & 0+0+1 & 0-1+1 \end{bmatrix}$
$= \begin{bmatrix} 2 & 2 & 3 \\ 4 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$

(c) $A^2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 1+0+0 & 0+0-1 & -1+0-1 \\ 2+0+0 & 0+0+1 & -2+0+1 \\ 0+2+0 & 0+0+1 & 0+1+1 \end{bmatrix}$
$= \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & -1 \\ 2 & 1 & 2 \end{bmatrix}$

(d) $B^2 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 0+2+0 & 0+1-2 & 0+0+2 \\ 0+2+0 & 2+1+0 & 4+0+0 \\ 0-2+0 & 0-1-1 & 0+0+1 \end{bmatrix}$
$= \begin{bmatrix} 2 & -1 & 2 \\ 2 & 3 & 4 \\ -2 & -2 & 1 \end{bmatrix}$

20. It's probably simplest to refer to 19(a) and 19(b) above.

21. (a) $(AB)C = \begin{bmatrix} 3 & -1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 4 & 5 \\ -2 & -7 \end{bmatrix}$
$A(BC) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 1 & -1 \end{bmatrix}$
$= \begin{bmatrix} 4 & 5 \\ -2 & -7 \end{bmatrix}$

(b) $(AB)C = \begin{bmatrix} 5 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 4 & 5 \end{bmatrix}$
$A(BC) = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$
$= \begin{bmatrix} 4 & 5 \end{bmatrix}$

22. (a) $A(B+C) = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$
$= \begin{bmatrix} 1 & 8 \\ 4 & 8 \end{bmatrix}$
$AB + AC = \begin{bmatrix} -2 & 3 \\ -4 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 8 & 4 \end{bmatrix}$
$= \begin{bmatrix} 1 & 8 \\ 4 & 8 \end{bmatrix}$

(b) $$\text{A}(\text{B}+\text{C}) = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\text{AB}+\text{AC} = \begin{bmatrix} 6 \\ -7 \end{bmatrix} + \begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

23. $$(k\text{A})\text{B} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} kae+kbg & kag+kah \\ kce+kdg & kcg+kdh \end{bmatrix}$$

$$\text{A}(k\text{B}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} ke & kf \\ kg & kh \end{bmatrix} = \begin{bmatrix} kae+kbg & kag+kah \\ kce+kdg & kcg+kdh \end{bmatrix}$$

$$k(\text{AB}) = k\begin{bmatrix} ae+bg & ag+ah \\ ce+dg & cg+dh \end{bmatrix} = \begin{bmatrix} kae+kbg & kag+kah \\ kce+kdg & kcg+kdh \end{bmatrix}$$

$$\therefore \quad (k\text{A})\text{B} = \text{A}(k\text{B}) = k(\text{AB})$$

□

24. No working required.

25. No working required. (Write down the dimensions of each matrix, then this question becomes a repeat of the previous one.)

26. No working required.

27. Consider each possible permutation of two matrices:

	A	B	C
A	AA	AB	AC
B	BA	BB	BC
C	CA	CB	CC

then simply decide which products have dimensions that allow multiplication:

	A	B	C
A	$(2\times2)(2\times2)$	~~$(2\times2)(1\times2)$~~	$(2\times2)(2\times1)$
B	$(1\times2)(2\times2)$	~~$(1\times2)(1\times2)$~~	$(1\times2)(2\times1)$
C	~~$(2\times1)(2\times2)$~~	$(2\times1)(1\times2)$	~~$(2\times1)(2\times1)$~~

Note: BC *is* a valid product even though not listed in Mr Sadler's solution. (It results in a 1×1 matrix.)

28. (a) $$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 4 & 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 7 & -3 \end{bmatrix}$$

29. (a) $$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 3 & 3 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \\ 12 \\ 11 \\ 13 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 3 & 3 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 15 \\ 10 \\ 14 \end{bmatrix}$$

30. Initially:

$$\begin{bmatrix} 1\,000 & 5\,000 & 400 & 270 \\ 500 & 8\,000 & 500 & 250 \\ 500 & 3\,000 & 500 & 500 \end{bmatrix}\begin{bmatrix} 5 \\ 0.5 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 15\,000 \\ 15\,000 \\ 15\,000 \end{bmatrix}$$

All client portfolios are initially worth \$15 000.

Two years later:

$$\begin{bmatrix} 1\,000 & 5\,000 & 400 & 270 \\ 500 & 8\,000 & 500 & 250 \\ 500 & 3\,000 & 500 & 500 \end{bmatrix}\begin{bmatrix} 4 \\ 0.6 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 17\,700 \\ 19\,300 \\ 18\,800 \end{bmatrix}$$

The portfolios of Client 1, Client 2 and Client 3 are worth \$17 700, \$19 300 and \$18 800 respectively.

31. $$\begin{bmatrix} 15 & 10 \end{bmatrix}\begin{bmatrix} 375 & 1 \\ 1250 & 4 \end{bmatrix} = \begin{bmatrix} 18\,125 & 55 \end{bmatrix}$$

The order requires 18 125mL of drink and 55 burgers.

32. (a) P is 3×3 and Q is 1×3 so QP is possible and PQ is not.

(b) $$\text{QP} = \begin{bmatrix} 75 & 125 & 180 \end{bmatrix}\begin{bmatrix} 15 & 5 & 5 \\ 25 & 25 & 14 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4\,610 & 3\,680 & 2\,665 \end{bmatrix}$$

The product shows the income per night for each hotel when all rooms are in use.

(c) Refer to the answer in Sadler.

33. No working required.

34. No working required.

35. Remember that the multiplication must make sense. We need to multiply the times for model A (i.e. the first row of P) with the number of orders for A; it makes no sense to multiply any of the times for A with the number of orders for B. PQ has as its first element (cutting A)×(orders A)+(assembling A)×(orders B)+(packing A)×(orders C) and thus makes no sense. In contrast, RP has as its first element (orders A)×(cutting A)+(orders B)×(cutting B)+(orders C)×(cutting C): the models are not mixed and the product makes sense. This should also make it clear what this first element is: the total cutting time. The meaning of the remainder of the RP should be obvious.

Exercise 3C

1–8 No working required.

9. $\dfrac{1}{2\times 1-1\times 1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

10. $\dfrac{1}{3\times 3-2\times 4}\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$

11. $\dfrac{1}{2\times 1-1\times -1}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

12. $\dfrac{1}{4\times 2-3\times 1}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$

13. $\dfrac{1}{3\times 3+1\times 1}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

14. $\dfrac{1}{9+1}\begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}$

15. Matrix is singular and so has no inverse.

16. Matrix is singular and so has no inverse.

17. Matrix is singular and so has no inverse.

18. $\dfrac{1}{x\times 1-y\times 0}\begin{bmatrix} 1 & -y \\ 0 & x \end{bmatrix} = \dfrac{1}{x}\begin{bmatrix} 1 & -y \\ 0 & x \end{bmatrix}$

19. $\dfrac{1}{1-0}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

20. $\dfrac{1}{-1-0}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -1\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

21. No working required.

22.
$$\begin{aligned}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\mathrm{A} &= \begin{bmatrix} 2 & 11 \\ 1 & 6 \end{bmatrix}\\ \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\mathrm{A} &= \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 11 \\ 1 & 6 \end{bmatrix}\\ \mathrm{A} &= \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\end{aligned}$$

23.
$$\mathrm{B}\begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix} \& = \begin{bmatrix} 7 & 1 \\ 15 & 3 \end{bmatrix}$$
$$\mathrm{B}\begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 15 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -3 & 5 \end{bmatrix}$$
$$\mathrm{B} = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$$

24.
$$\begin{aligned}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\mathrm{C} &= \begin{bmatrix} -1 & 4 & -2 \\ 1 & -1 & 1 \end{bmatrix}\\ -1\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\mathrm{C} &= -1\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 & -2 \\ 1 & -1 & 1 \end{bmatrix}\\ \mathrm{C} &= -1\begin{bmatrix} -1 & -2 & 0 \\ 1 & -1 & 1 \end{bmatrix}\\ &= \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & -1 \end{bmatrix}\end{aligned}$$

25.
$$\mathrm{D}\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix} \& = \begin{bmatrix} -2 & -6 \end{bmatrix}$$
$$\mathrm{D}\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}\frac{1}{10}\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -6 \end{bmatrix}\frac{1}{10}\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$
$$\begin{aligned}\mathrm{D} &= \frac{1}{10}\begin{bmatrix} 10 & -30 \end{bmatrix}\\ &= \begin{bmatrix} 1 & -3 \end{bmatrix}\end{aligned}$$

26.
$$\begin{aligned}\mathrm{A}^2 - 2\mathrm{A} &= \begin{bmatrix} 13 & 8 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 2 & -2 \end{bmatrix}\\ &= \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\\ &= 7\mathrm{I}\\ \therefore\quad k &= 7\end{aligned}$$

27.
$$\begin{aligned}\mathrm{A}^{-1} &= \frac{1}{0k+10}\begin{bmatrix} 0 & 2 \\ -5 & k \end{bmatrix}\\ \therefore\quad 10\mathrm{A}^{-1} &= \begin{bmatrix} 0 & 2 \\ -5 & k \end{bmatrix}\\ \mathrm{A} + 10\mathrm{A}^{-1} &= \begin{bmatrix} k & -2 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -5 & k \end{bmatrix}\\ &= \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\\ &= k\mathrm{I}\\ \therefore\quad k &= 5\end{aligned}$$

28. (a) No working required.

(b) $16\times 5 + 5\times -14 = 10$

(c)
$$\begin{aligned}\mathrm{A}^{-1} &= \frac{1}{3-2}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\\ &= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\end{aligned}$$

(d) No working required (since we've already calculated the determinant of B).

(e)
$$\begin{aligned}\mathrm{AC} &= \mathrm{B}\\ \mathrm{A}^{-1}\mathrm{AC} &= \mathrm{A}^{-1}\mathrm{B}\\ \mathrm{C} &= \mathrm{A}^{-1}\mathrm{B}\\ &= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 16 & -5 \\ -14 & 5 \end{bmatrix}\\ &= \begin{bmatrix} 2 & 0 \\ -10 & 5 \end{bmatrix}\end{aligned}$$

(f)
$$\begin{aligned}\mathrm{DA} &= \mathrm{B}\\ \mathrm{DAA}^{-1} &= \mathrm{BA}^{-1}\\ \mathrm{D} &= \mathrm{BA}^{-1}\\ &= \begin{bmatrix} 16 & -5 \\ -14 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\\ &= \begin{bmatrix} 6 & 1 \\ -4 & 1 \end{bmatrix}\end{aligned}$$

29. (a) No working required.

(b)
$$\begin{aligned}\mathrm{P}^{-1} &= \frac{1}{-4+3}\begin{bmatrix} 1 & -1 \\ -3 & -4 \end{bmatrix}\\ &= \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}\end{aligned}$$

(c)
$$\begin{aligned}\mathrm{Q}^{-1} &= \frac{1}{6-0}\begin{bmatrix} 1 & -2 \\ 0 & 6 \end{bmatrix}\\ &= \frac{1}{6}\begin{bmatrix} 1 & -2 \\ 0 & 6 \end{bmatrix}\end{aligned}$$

(d) $(P+Q)^{-1} = \frac{1}{4-3}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

(e)
$$\begin{aligned} R(P+Q) &= Q \\ R(P+Q)(P+Q)^{-1} &= Q(P+Q)^{-1} \\ R &= Q(P+Q)^{-1} \\ &= \begin{bmatrix} 6 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix} \end{aligned}$$

30. $A = (AB)B^{-1}$
$= \begin{bmatrix} 2 & -1 \\ 17 & -9 \end{bmatrix}$

31. $D = C^{-1}(CD)$
$= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$

32. (a) $3x - 24 = 0$
$x = 8$

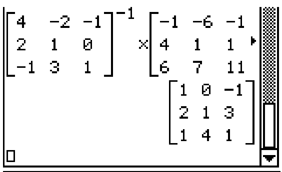
(b) $x^2 - 16 = 0$
$x = \pm 4$

(c)
$$\begin{aligned} x^2 - x - 20 &= 0 \\ (x-5)(x+4) &= 0 \\ x &= 5 \\ \text{or} \quad x &= -4 \end{aligned}$$

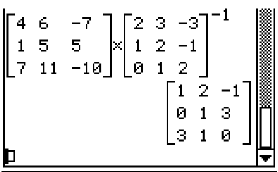
33. $F = E^{-1}(EF)$
$= \frac{1}{2}\begin{bmatrix} 2 & 6 \\ 4 & -6 \end{bmatrix}$
$= \begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix}$

$G = (GE)E^{-1}$
$= \frac{1}{2}\begin{bmatrix} 6 & -8 \\ 0 & 4 \end{bmatrix}$
$= \begin{bmatrix} 3 & -4 \\ 0 & 2 \end{bmatrix}$

34. $AC = B$
$C = A^{-1}B$



35. $CA = B$
$C = BA^{-1}$



36. (a) No working required.

(b)
$$\begin{aligned} CA &= B \\ C &= BA^{-1} \\ &= \begin{bmatrix} 24 & 56 \\ 16 & 36 \end{bmatrix}\frac{1}{30\times 36 - 70\times 16}\begin{bmatrix} 36 & -70 \\ -16 & 30 \end{bmatrix} \\ &= -\frac{1}{40}\begin{bmatrix} 24 & 56 \\ 16 & 36 \end{bmatrix}\begin{bmatrix} 36 & -70 \\ -16 & 30 \end{bmatrix} \\ &= -\frac{1}{40}\begin{bmatrix} -32 & 0 \\ 0 & -40 \end{bmatrix} \\ &= \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

37.
$$\begin{aligned} C &= A - CB \\ C + CB &= A \\ CI + CB &= A \\ C(I+B) &= A \\ C(I+B)(I+B)^{-1} &= A(I+B)^{-1} \\ C &= A(I+B)^{-1} \end{aligned}$$

□

$$\begin{aligned} C &= A(I+B)^{-1} \\ &= \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -5 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}\frac{1}{14}\begin{bmatrix} 2 & -2 \\ 5 & 2 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 28 & 14 \\ 42 & -14 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \end{aligned}$$

38.
$$\begin{aligned} A &= BC - AC \\ &= (B-A)C \\ (B-A)^{-1}A &= (B-A)^{-1}(B-A)C \\ C &= (B-A)^{-1}A \\ B - A &= \begin{bmatrix} -1 & 0 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 5 \\ 1 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \\ (B-A)^{-1} &= \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix} \\ C &= \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 1 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 20 \\ 5 & 7 \end{bmatrix} \end{aligned}$$

39.
$$\begin{aligned}
P - PQ - PQ^2 &= Q\\
P(I - Q - Q^2) &= Q\\
P &= Q(I - Q - Q^2)^{-1}\\
Q^2 &= \begin{bmatrix} 1 & 0 \\ -15 & 4 \end{bmatrix}\\
I - Q - Q^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -15 & 4 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 \\ 10 & -1 \end{bmatrix}\\
(I - Q - Q^2)^{-1} &= -1\begin{bmatrix} -1 & 0 \\ -10 & 1 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 \\ 10 & -1 \end{bmatrix}\\
P &= \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 10 & -1 \end{bmatrix}\\
&= \begin{bmatrix} -1 & 0 \\ -15 & 2 \end{bmatrix}
\end{aligned}$$

40. (a) No working required.

(b) No working required.

(c)
$$\begin{aligned}
BA &= \begin{bmatrix} 860 & 740 \end{bmatrix}\\
BAA^{-1} &= \begin{bmatrix} 860 & 740 \end{bmatrix} A^{-1}\\
B &= \begin{bmatrix} 860 & 740 \end{bmatrix}\begin{bmatrix} 6 & 5 \\ 8 & 7 \end{bmatrix}^{-1}\\
&= \begin{bmatrix} 860 & 740 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 7 & -5 \\ -8 & 6 \end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} 50 & 70 \end{bmatrix}\\
\begin{bmatrix} x & y \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 50 & 70 \end{bmatrix}\\
x &= 50\\
y &= 70
\end{aligned}$$

Exercise 3D

1–6 No working required.

7. (a) No working required. (By this stage you should be able to find the inverse of a 2×2 matrix in a single step.)

(b)
$$\begin{aligned}
\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 4 \\ -9 \end{bmatrix}\\
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ -9 \end{bmatrix}\\
&= \begin{bmatrix} -1 \\ -\frac{7}{2} \end{bmatrix}\\
x &= -1\\
y &= -\frac{7}{2}
\end{aligned}$$

8. (a) No working required.

(b)
$$\begin{aligned}
\begin{bmatrix} -2 & 1 & -2 \\ 2 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 3 \\ -1 \\ 9 \end{bmatrix}\\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} -5 & -4 & 1 \\ -4 & -4 & 2 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 9 \end{bmatrix}\\
&= \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}\\
x &= -1\\
y &= 5\\
z &= 2
\end{aligned}$$

9. (a)
$$\begin{aligned}
\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}\\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\\
&= \begin{bmatrix} 3 \\ -7 \end{bmatrix}\\
x &= 3\\
y &= -7
\end{aligned}$$

(b)
$$\begin{aligned}
\begin{bmatrix} 3 & 1 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 8 \\ 13 \end{bmatrix}\\
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 13 \end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} 11 \\ -17 \end{bmatrix}\\
x &= 5.5\\
y &= -8.5
\end{aligned}$$

10. (a)
$$\begin{aligned}
AB &= \begin{bmatrix} 8 - 11 + 10 & -4 + 2 + 2 & 2 - 8 + 6 \\ -4 - 11 + 15 & 2 + 2 + 3 & -1 - 8 + 9 \\ -12 + 22 - 10 & 6 - 4 - 2 & -3 + 16 - 6 \end{bmatrix}\\
&= \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\\
&= 7I
\end{aligned}$$

(b)
$$\begin{aligned}
7I &= AB\\
I &= \frac{1}{7}AB\\
A^{-1}I &= A^{-1}\frac{1}{7}AB\\
A^{-1} &= \frac{1}{7}B
\end{aligned}$$

(c)
$$\mathrm{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathrm{A}^{-1}\begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix}$$
$$= \frac{1}{7}\mathrm{B}\begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix}$$
$$= \frac{1}{7}\begin{bmatrix} 12+14-5 \\ -33-14+40 \\ -15+7+15 \end{bmatrix}$$
$$= \frac{1}{7}\begin{bmatrix} 21 \\ -7 \\ 7 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$
$$x = 3$$
$$y = -1$$
$$z = 1$$

11. No working required for (a) and (b) is purely calculator work to determine $\mathrm{X} = \mathrm{A}^{-1}\mathrm{B}$, then interpret the result.

Miscellaneous Exercise 3

1. No working required.

2. (a) $2z = 2 \times 3 \operatorname{cis} \frac{5\pi}{6} = 6 \operatorname{cis} \frac{5\pi}{6}$

   (b) $3w = 3 \times 2 \operatorname{cis}\left(-\frac{2\pi}{3}\right) = 6 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

   (c) $zw = 3 \times 2 \operatorname{cis}\left(\frac{5\pi}{6} - \frac{2\pi}{3}\right) = 6 \operatorname{cis} \frac{\pi}{6}$

   (d)
$$\frac{z}{w} = \frac{3}{2} \operatorname{cis}\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right)$$
$$= 1.5 \operatorname{cis} \frac{9\pi}{6}$$
$$= 1.5 \operatorname{cis}\left(\frac{3\pi}{2} - 2\pi\right)$$
$$= 1.5 \operatorname{cis}\left(-\frac{\pi}{2}\right)$$

   (e)
$$\mathrm{i}z = 3 \operatorname{cis}\left(\frac{5\pi}{6} + \frac{\pi}{2}\right)$$
$$= 3 \operatorname{cis} \frac{8\pi}{6}$$
$$= 3 \operatorname{cis}\left(\frac{4\pi}{3} - 2\pi\right)$$
$$= 3 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

   (f) $-w = 2 \operatorname{cis}\left(-\frac{2\pi}{3} + \pi\right) = 2 \operatorname{cis} \frac{\pi}{3}$

   (g) No working required.

   (h) No working required (using the answer to (c) as the starting point).

   (i) No working required (because $\bar{z}\bar{w} = \overline{zw}$).

   (j)
$$z^2w^3 = 3^2 \operatorname{cis}\left(\frac{5\pi}{6} \times 2\right) \times 2^3 \operatorname{cis}\left(-\frac{2\pi}{3} \times 3\right)$$
$$= 9 \times 8 \operatorname{cis}\left(\frac{5\pi}{3} - 2\pi\right)$$
$$= 72 \operatorname{cis}\left(-\frac{\pi}{3}\right)$$

   It is useful to remember that

   - $\mathrm{i} = \operatorname{cis} \frac{\pi}{2}$
   - $-1 = \operatorname{cis} \pi = \operatorname{cis}(-\pi)$
   - $-\mathrm{i} = \operatorname{cis}\left(-\frac{\pi}{2}\right)$

3. (a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \end{bmatrix}$

4. Write the dimensions for each matrix and rearrange so that adjacent numbers match:
   B A C
   1×2 2×3 3×4
   Before doing any calculations, it should be clear

that this will result in a 1×4 matrix.

$$\begin{aligned}\mathrm{BA} &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \\ \mathrm{BAC} &= \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 3 \\ 3 & 1 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 5 & 2 & 4 \end{bmatrix}\end{aligned}$$

5. $$\begin{aligned}A &= \int_3^6 x^2 - 2x + 3\,\mathrm{d}x \\ &= \left[\frac{x^3}{3} - x^2 + 3x\right]_3^6 \\ &= (72 - 36 + 18) - (9 - 9 + 9) \\ &= 54 - 9 \\ &= 45 \text{ units}^2\end{aligned}$$

6. (a) $$\begin{aligned}\text{LHS} &= \mathrm{e}^{\mathrm{i}(\alpha+\beta)} \\ &= \mathrm{e}^{\mathrm{i}\alpha + \mathrm{i}\beta} \\ &= \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta} \\ &= \operatorname{cis}\alpha \operatorname{cis}\beta \\ &= \text{RHS}\end{aligned}$$

□

(b) $$\begin{aligned}\text{LHS} &= \cos(\alpha+\beta) + \mathrm{i}\sin(\alpha+\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &\quad + \mathrm{i}(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &\quad + \mathrm{i}\sin\alpha\cos\beta + \mathrm{i}\cos\alpha\sin\beta \\ &= \cos\alpha\cos\beta + \mathrm{i}\cos\alpha\sin\beta \\ &\quad - \sin\alpha\sin\beta + \mathrm{i}\sin\alpha\cos\beta \\ &= \cos\alpha(\cos\beta + \mathrm{i}\sin\beta) \\ &\quad + \mathrm{i}^2\sin\alpha\sin\beta + \mathrm{i}\sin\alpha\cos\beta \\ &= \cos\alpha(\cos\beta + \mathrm{i}\sin\beta) \\ &\quad + \mathrm{i}\sin\alpha(\mathrm{i}\sin\beta + \cos\beta) \\ &= (\cos\alpha + \mathrm{i}\sin\alpha)(\cos\beta + \mathrm{i}\sin\beta) \\ &= \operatorname{cis}\alpha\operatorname{cis}\beta \\ &= \text{RHS}\end{aligned}$$

□

7. No working required.
(For (c), think $4\mathrm{e}^{\mathrm{i}\pi x} = 4\operatorname{cis}(\pi x)$ so the set described is the points $z = r\operatorname{cis}\theta$ having modulus $r = 4$ and argument $0 < \theta \leq \frac{\pi}{2}$.)

8. $$\begin{aligned}A &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 2\sin^2 x\,\mathrm{d}x \\ &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} -(1 - 2\sin^2 x) + 1\,\mathrm{d}x \\ &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} -\cos 2x + 1\,\mathrm{d}x \\ &= \left[-\frac{\sin 2x}{2} + x\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \\ &= \left(-\frac{\sin\frac{4\pi}{3}}{2} + \frac{2\pi}{3}\right) - \left(-\frac{\sin\frac{2\pi}{3}}{2} + \frac{\pi}{3}\right) \\ &= -\frac{-\frac{\sqrt{3}}{2}}{2} + \frac{\frac{\sqrt{3}}{2}}{2} + \frac{2\pi}{3} - \frac{\pi}{3} \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + \frac{\pi}{3} \\ &= \frac{\sqrt{3}}{2} + \frac{\pi}{3} \text{ units}^2\end{aligned}$$

9. $$\begin{aligned}r &= \sqrt{3+1} \\ &= 2 \\ \tan\theta &= \frac{1}{-\sqrt{3}} \qquad \text{(second quadrant)} \\ \theta &= \frac{5\pi}{6} \\ \therefore\quad -\sqrt{3} + \mathrm{i} &= 2\operatorname{cis}\frac{5\pi}{6} \\ \left(-\sqrt{3} + \mathrm{i}\right)^{12} &= 2^{12}\operatorname{cis}\left(\frac{5\pi}{6} \times 12\right) \\ &= 4096\operatorname{cis}(10\pi) \\ &= 4096\operatorname{cis}0 \\ &= 4096\end{aligned}$$

10. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x - 1 \\ &= 11 \\ y - y_1 &= m(x - x_1) \\ y - 5 &= 11(x - 2) \\ y &= 11x - 17\end{aligned}$$

11. $$\begin{aligned}2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= -2x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y} \\ &= \frac{3}{4} \\ y - y_1 &= m(x - x_1) \\ y + 4 &= \frac{3}{4}(x - 3) \\ 4y + 16 &= 3x - 9 \\ 3x - 4y &= 25\end{aligned}$$

12. Points where $x = -2$:

$$\begin{aligned} -2y + y^2 - (-2)^3 &= 11 \\ y^2 - 2y + 8 &= 11 \\ y^2 - 2y - 3 &= 0 \\ (y-3)(y+1) &= 0 \end{aligned}$$

The points are $(-2, -1)$ and $(-2, 3)$.
Differentiating:

$$\begin{aligned} y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2 &= 0 \\ x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 - y \\ \frac{\mathrm{d}y}{\mathrm{d}x}(x + 2y) &= 3x^2 - y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2 - y}{x + 2y} \end{aligned}$$

At $(-2, -1)$:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{12 + 1}{-2 - 2} \\ &= -\frac{13}{4} \\ y - y_1 &= m(x - x_1) \\ y + 1 &= -\frac{13}{4}(x + 2) \\ 4y + 4 &= -13x - 26 \\ 13x + 4y &= -30 \end{aligned}$$

At $(-2, 3)$:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{12 - 3}{-2 + 6} \\ &= \frac{9}{4} \\ y - y_1 &= m(x - x_1) \\ y - 3 &= \frac{9}{4}(x + 2) \\ 4y - 12 &= 9x + 18 \\ 9x - 4y &= -30 \end{aligned}$$

13. (a) The column matrix Y is useful. When forming the product XY, the number of units of the commodities is multiplied by the cost of the corresponding commodity.

(b) $$\begin{aligned} \mathrm{XY} &= \begin{bmatrix} 100 + 120 + 200 \\ 150 + 60 + 200 \\ 50 + 180 + 200 \end{bmatrix} \\ &= \begin{bmatrix} 420 \\ 410 \\ 430 \end{bmatrix} \end{aligned}$$

(c) The product gives the total cost for each of the three models.

14. The initial case, where $n = 1$:

$$\begin{aligned} \text{R.H.S.} &= \frac{r(r^1 - 1)}{r - 1} \\ &= r \\ &= \text{L.H.S.} \end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$r + r^2 + r^3 + \ldots + r^k = \frac{r(r^k - 1)}{r - 1}$$

Then for $n = k + 1$

$$\begin{aligned} \text{L.H.S.} &= r + r^2 + r^3 + \ldots + r^k + r^{k+1} \\ &= \frac{r(r^k - 1)}{r - 1} + r^{k+1} \\ &= \frac{r(r^k - 1)}{r - 1} + \frac{r^{k+1}(r - 1)}{r - 1} \\ &= \frac{r(r^k - 1)}{r - 1} + \frac{rr^k(r - 1)}{r - 1} \\ &= \frac{r(r^k - 1)}{r - 1} + \frac{r(r^{k+1} - r^k)}{r - 1} \\ &= \frac{r(r^k - 1 + r^{k+1} - r^k)}{r - 1} \\ &= \frac{r(r^{k+1} - 1)}{r - 1} \\ &= \text{R.H.S.} \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

15. For the matrix to be singular, the determinant must be zero. For this matrix the determinant is $2x^2 + 4$. This quadratic has no real roots, so the matrix cannot be singular for $x \in \Re$.

16.
$$\begin{aligned} \mathrm{A}^2 &= \begin{bmatrix} k & 4 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} k & 4 \\ -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} k^2 - 12 & 4k - 4 \\ -3k + 3 & -11 \end{bmatrix} \\ \mathrm{A}^2 + \mathrm{A} &= \begin{bmatrix} k^2 - 12 & 4k - 4 \\ -3k + 3 & -11 \end{bmatrix} + \begin{bmatrix} k & 4 \\ -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} k^2 + k - 12 & 4k \\ -3k & -12 \end{bmatrix} \\ \begin{bmatrix} 0 & p \\ q & -12 \end{bmatrix} &= \begin{bmatrix} k^2 + k - 12 & 4k \\ -3k & -12 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} k^2 + k - 12 &= 0 \\ (k + 4)(k - 3) &= 0 \\ p &= 4k \\ q &= -3k \end{aligned}$$

$$\begin{aligned} k &= -4 \\ p &= -16 \\ q &= 12 \end{aligned}$$

$$\begin{aligned} \text{or } k &= 3 \\ p &= 12 \\ q &= -9 \end{aligned}$$

Reject the first solution set because it does not satisfy $p > 0$ and conclude

$$\begin{aligned} k &= 3 \\ p &= 12 \\ q &= -9 \end{aligned}$$

17. (a)
$$\begin{aligned} \frac{\ln x}{x} &= 0 \\ \ln x &= 0 \\ x &= 1 \end{aligned}$$

The coordinates of A are $(1, 0)$.

(b)
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} \\ &= \frac{1 - \ln x}{x^2} \end{aligned}$$

At B,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{1 - \ln x}{x^2} &= 0 \\ 1 - \ln x &= 0 \\ \ln x &= 1 \\ x &= \mathrm{e} \\ y &= \frac{\ln x}{x} \\ &= \frac{\ln \mathrm{e}}{\mathrm{e}} \\ &= \frac{1}{\mathrm{e}} \end{aligned}$$

The coordinates of B are $(\mathrm{e}, \frac{1}{\mathrm{e}})$

(c)
$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\left(-\frac{1}{x}\right)(x^2) - (1 - \ln x)(2x)}{x^4} \\ &= \frac{-x - 2x + 2x\ln x}{x^4} \\ &= \frac{2x\ln x - 3x}{x^4} \\ &= \frac{2\ln x - 3}{x^3} \end{aligned}$$

At C,

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 0 \\ \frac{2\ln x - 3}{x^3} &= 0 \\ 2\ln x - 3 &= 0 \\ \ln x &= 1.5 \\ x &= \mathrm{e}^{1.5} \\ y &= \frac{\ln x}{x} \\ &= \frac{1.5}{\mathrm{e}^1.5} \\ &= 1.5\mathrm{e}^{-1.5} \end{aligned}$$

The coordinates of C are $(\mathrm{e}^{1.5}, 1.5\mathrm{e}^{-1.5})$

18. Because of the symmetry, we need only consider one quadrant. The positive $x-$intercept is at

$$\begin{aligned} 8x - 16 &= 0 \\ x &= 2. \end{aligned}$$

Rewriting the right bound in the first quadrant as a function of x gives

$$y = \sqrt{8x - 16}$$

The $x-$coordinate of the top-right point is given by the intersection of the top and right curves:

$$\begin{aligned} \sqrt{8x - 16} &= 2 + \frac{1}{8}x^2 \\ 8x - 16 &= 4 + \frac{1}{2}x^2 + \frac{1}{64}x^4 \end{aligned}$$

This looks messy to solve, but there's a simpler approach: based on the symmetry we know that this point also intersects the line $y = x$, giving us

$$\begin{aligned} 2 + \frac{1}{8}x^2 &= x \\ 16 + x^2 &= 8x \\ x^2 - 8x + 16 &= 0 \\ (x - 4)^2 &= 0 \\ x &= 4 \end{aligned}$$

The area in the first quadrant is given by

$$A = \int_0^4 2 + \frac{1}{8}x^2 \,\mathrm{d}x - \int_2^4 \sqrt{8x - 16} \,\mathrm{d}x$$

Although we should be able to calculate this, there is again a simpler approach. Rather than use the square root function for the lower bound of the region we will find only the area above the line $y = x$. Based on the symmetry, we know that this will give us half the area of the first quadrant (or one eighth of the total area).

$$\begin{aligned} A &= 2\int_0^4 2 + \frac{1}{8}x^2 - x \,\mathrm{d}x \\ &= \int_0^4 4 + \frac{1}{4}x^2 - 2x \,\mathrm{d}x \\ &= \left[4x + \frac{x^3}{12} - x^2\right]_0^4 \\ &= \left[x(4 + \frac{x^2}{12} - x)\right]_0^4 \\ &= 4(4 + \frac{4^2}{12} - 4) - 0 \\ &= \frac{16}{3} \end{aligned}$$

Thus the total area is $\frac{64}{3} \approx 21.33\text{cm}^2$.

(Note the correct units for this answer are square centimetres, not square units as shown in Sadler.)

It should be noted that questions like this have several different paths to the correct answer, some much simpler than others. You should always be on the lookout for a simpler approach, even if it means changing track part way through the problem as I have done here.

Another approach to this problem, possibly simpler still, would be to consider the area as an 8×8 square with four parabolic 'bites' taken out of it and determine the area of each of these bites as

$$\begin{aligned}A &= \int_{-4}^{4} 4-(2+\frac{1}{8}x^2)\,\mathrm{d}x\\ &= \int_{-4}^{4} 2-\frac{1}{8}x^2\,\mathrm{d}x\\ &= \left[2x-\frac{x^3}{24}\right]_{-4}^{4}\\ &= \left(8-\frac{16}{6}\right)-\left(-8+\frac{16}{6}\right)\\ &= 16-\frac{16}{3}\\ \text{Total area} &= 8^2-4\left(16-\frac{16}{3}\right)\\ &= \frac{64}{3}\end{aligned}$$

If I were preparing this as a "display answer", I would include only the determination of the points of intersection and then this last approach. I leave the other work in this solution simply to show students the kind of thinking process that a capable student would go through for a complex problem like this.

19. (a) The curve intersects the $x-$axis in the given domain at $x=0$ and $x=\frac{\pi}{2}$. The area enclosed by the curve and the axis is then (using the calculator)

$$\begin{aligned}A &= \int_{0}^{\frac{\pi}{2}} |3x^2(1-\sin x)|\\ &= 0.451 \quad (3\text{ d.p.})\end{aligned}$$

But this is not the exact value we require, so we'll use the calculator to give us the *indefinite* integral, then substitute to get the exact area:

$$\begin{aligned}A &= \left[x^3+3x^2\cos x-6x\sin x-6\cos x\right]_0^{\frac{\pi}{2}}\\ &= \left(\frac{\pi^3}{8}-3\pi\right)-(-6)\\ &= \frac{\pi^3}{8}-3\pi+6\end{aligned}$$

(b) The curve intersects the line in the given domain where

$$\frac{25\pi(2x-\pi)}{1}6 = 3x^2(1-\sin x)$$

for which the calculator gives solutions at $x=1.571$ and $x=2.618$. However, these are not the exact values we will need for the bounds, so it needs further work. The first solution looks like $\frac{\pi}{2}$ which we can confirm:

$$\begin{aligned}\frac{25\pi(2x-\pi)}{1}6 &= 3x^2(1-\sin x)\\ \frac{25\pi\left(2\left(\frac{\pi}{2}\right)\pi-\pi\right)}{1}6 &= 0\\ 3\left(\frac{\pi}{2}\right)^2\left(1-\sin\frac{\pi}{2}\right) &= 0\end{aligned}$$

Guessing that the second intercept is also a multiple of π we obtain $2.618\approx0.8333\pi=\frac{5\pi}{6}$. We can also confirm this:

$$\begin{aligned}\frac{25\pi(2x-\pi)}{1}6 &= 3x^2(1-\sin x)\\ \frac{25\pi\left(2\left(\frac{5\pi}{6}\right)\pi-\pi\right)}{1}6 &= \frac{25\pi^2}{24}\\ 3\left(\frac{5\pi}{6}\right)^2\left(1-\sin\frac{5\pi}{6}\right) &= \frac{25\pi^2}{24}\end{aligned}$$

The area enclosed by the curve and the line is then

$$\begin{aligned}A &= \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \left|\frac{25\pi(2x-\pi)}{1}6-3x^2(1-\sin x)\right|\\ &= 2.355 \quad (3\text{ d.p.})\end{aligned}$$

Again, this is not the exact value we want so we'll again use the calculator to obtain an indefinite ingtegral and proceed from there.

$$\begin{aligned}A &= \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \frac{25\pi(2x-\pi)}{1}6-3x^2(1-\sin x)\\ &= \left[\begin{array}{c}\frac{-16x^3+46x^2\cos(x)-25x^2\pi}{16}\\ +\frac{-96x\sin x-96\cos x+25x\pi^2}{16}\end{array}\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}\\ &= \frac{-\left(\frac{1375\pi^3}{108}-40\pi-\frac{50\sqrt{3}\pi^2}{3}+48\sqrt{3}\right)}{16}\\ &\quad+\frac{\frac{33\pi^3}{4}-48\pi}{16}\\ &= -\frac{121\pi^3}{432}-\frac{\pi}{2}+\frac{25\sqrt{3}\pi^2}{24}-3\sqrt{3}\end{aligned}$$

This is a challenging calculator task. Calculator skills involved: graphing, solving equations, finding definite and indefinite integrals, evaluating an expression in x given a particular value of x, storing intermediate expressions in variables, defining functions, etc.