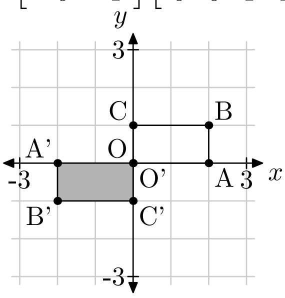
Chapter 4

Exercise 4A

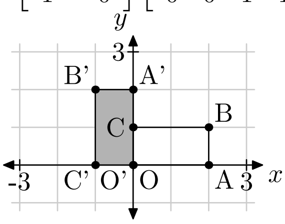
1–12 For these questions, rather than pre-multiply each of O, A, B and C by the given matrix, I will assemble [OABC] into a 2 × 4 matrix $\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and do the matrix multiplication in a single step.

1. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$



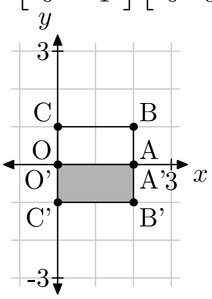
This represents a 180° rotation.

2. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 2 & 2 & 0 \end{bmatrix}$



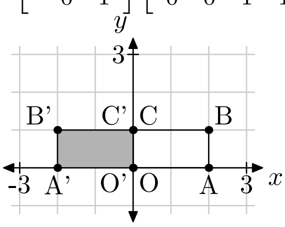
This represents a 90° anticlockwise rotation.

3. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$



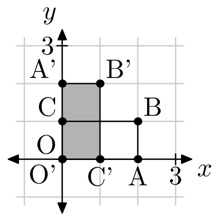
This represents a reflection in the $x-$axis.

4. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$



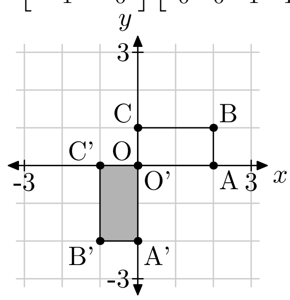
This represents a reflection in the $y-$axis.

5. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{bmatrix}$



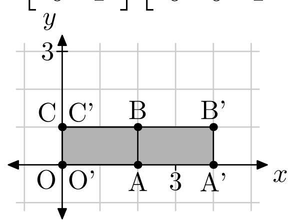
This represents a reflection in the line $y = x$.

6. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 0 \end{bmatrix}$



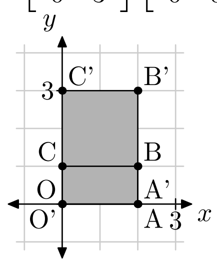
This represents a reflection in the line $y = -x$.

7. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$



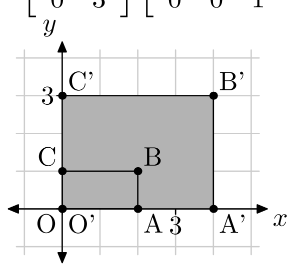
This represents a horizontal dilation of factor 2.

8. $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$



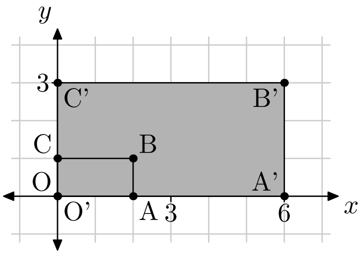
This represents a vertical dilation of factor 3.

9. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$



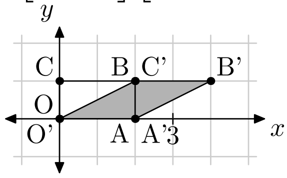
This represents a dilation with a horizontal scale factor of 2 and vertical scale factor of 3.

10. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 6 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$



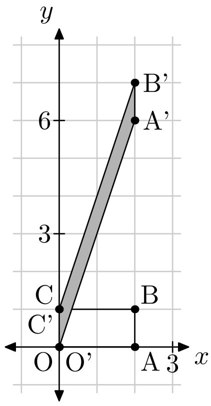
This represents a dilation with uniform scale factor of 3.

11. $$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$



This represents a shear parallel to the $x-$axis with scale factor of 2.

12. $$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 6 & 7 & 1 \end{bmatrix}$$



This represents a shear parallel to the $y-$axis with scale factor of 3.

13. The working needed here is quite straightforward. I present a worked solution for the first matrix only.

(a) $\det \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1 \times -1 + 0 \times 0 = 1$

(b)
$$\begin{aligned} \text{Area OABC} &= 2 \times 1 = 2 \\ \text{Area O'A'B'C'} &= 2 \times 1 = 2 \\ \frac{\text{Area O'A'B'C'}}{\text{Area OABC}} &= \frac{2}{2} = 1 \end{aligned}$$

Exercise 4B

1. (a) For matrix A, $(1,0)$ maps to $(0,-1)$ and $(0,1)$ maps to $(1,0)$; the required matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

For matrix B, $(1,0)$ maps to $(-1,0)$ and $(0,1)$ maps to $(0,-1)$; the required matrix is

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

For matrix C, $(1,0)$ maps to $(0,1)$ and $(0,1)$ maps to $(-1,0)$; the required matrix is

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(b)
$$\begin{aligned} A^2 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0^2 + 1 \times -1 & 0 \times 1 + 1 \times 0 \\ -1 \times 0 + 0 \times -1 & -1 \times 1 + 0^2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= B \end{aligned}$$

(c)
$$\begin{aligned} C^2 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0^2 - 1 \times 1 & 0 \times -1 - 1 \times 0 \\ 1 \times 0 + 0 \times 1 & 1 \times -1 + 0^2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= B \end{aligned}$$

(d)
$$\begin{aligned} A^3 &= A^2 A \\ &= BA \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 \times 0 + 0 \times -1 & -1 \times 1 + 0^2 \\ 0^2 + (-1)^2 & 0 \times 1 - 1 \times 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= C \end{aligned}$$

(e) $B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2$

$= \begin{bmatrix} (-1)^2 + 0^2 & -1 \times 0 + 0 \times -1 \\ 0 \times -1 - 1 \times 0 & 0^2 + (-1)^2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I$

(f) $A^{-1} = \dfrac{1}{0^2 - (-1 \times 1)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$= \dfrac{1}{1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$= C$

(Alternatively, show that $AC = I$)

(g) $B^{-1} = \dfrac{1}{(-1)^2 - 0^2} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$= \dfrac{1}{1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$= B$

Alternatively, since we have already shown that $B^2 = I$,

$$\begin{aligned} B^2 &= I \\ B^{-1}B^2 &= B^{-1}I \\ (B^{-1}B)B &= B^{-1} \\ IB &= B^{-1} \\ B &= B^{-1} \end{aligned}$$

2. (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

The transformation matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The transformation matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

The transformation matrix is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(d) A reflection in the $x-$axis followed by a reflection in the $y-$axis is represented by premultiplying the matrix for the first reflection by the matrix for the second, i.e.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

A reflection in the $y-$axis followed by a reflection in the $x-$axis is represented by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(e) Compare the results from (d) and (e).

3. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

The transformation matrix is $P= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

If P is its own inverse, then $P^2 = I$.

$$\begin{aligned} P^2 &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

4. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The transformation matrix is $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

The determinant of this matrix is $3 \times 1 - 0 \times 0 = 3$ as expected.

5. (a) No working needed.

(b) No working needed. (A, B, C and D are the columns of the second matrix and A′, B′, C′ and D′ are the columns of the product.)

6.
$$\begin{aligned} TA &= A' \\ T^{-1}TA &= T^{-1}A' \\ A &= T^{-1}A' \end{aligned}$$

$$\begin{aligned} T^{-1} &= \frac{1}{1 \times 1 - 2 \times 0} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} C &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -3 \end{bmatrix} \end{aligned}$$

A, B and C have coordinates $(1, 3)$, $(1, 1)$ and $(4, -3)$ respectively.

7. $$\begin{aligned} T^{-1} &= \frac{1}{2\times 1 - 0\times -3}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \\ A &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2 \\ 6 \end{bmatrix} \\ B &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ C &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 0 \\ 4 \end{bmatrix} \end{aligned}$$
A, B and C have coordinates $(1, 3)$, $(-1, 2)$ and $(0, 2)$ respectively.

8. $$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P &= P' \\ \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} P' &= P'' \\ \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P &= P'' \\ \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} P &= P'' \end{aligned}$$
Matrix $\begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$ will transform PQR directly to P″Q″R″.

9. $$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}^{-1} &= \frac{1}{-1-0}\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \end{aligned}$$
Matrix $\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ will transform PQR directly to P″Q″R″.

Matrix $\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ will transform P″Q″R″ directly to PQR. (The matrix is its own inverse.)

10. A shear parallel to the y-axis, scale factor 3, transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

A clockwise rotation of 90° about the origin transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The single matrix to perform both these transformations in sequence is
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

11. These are the same transformations as in the previous question, simply applied in the opposite order, so the single matrix to perform both these transformations in this new sequence is
$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

12. Post-multiply both sides of the equation with the inverse of $\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$ to eliminate it from the LHS:
$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} &= \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} \\ \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} &= \frac{1}{1+6}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 14 & 35 \\ 7 & 21 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \end{aligned}$$
so $a = 2$, $b = 5$, $c = 1$ and $d = 3$.

13. (a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ -1 & -4 \end{bmatrix}$

(c) $$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} &= \frac{1}{1-0}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \end{aligned}$$

(d) First, to transform $A_3B_3C_3D_3$ to $A_2B_2C_2D_2$
$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} &= \frac{1}{0+1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

then to further transform the result to $A_1B_1C_1D_1$ we use the matrix we obtained in (c), so the single matrix that combines both is
$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

14. A reflection in the x-axis transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

    A reflection in the line $y = x$ transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

    A $90°$ clockwise rotation is represented by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (see question 10).

    The matrix that represents these three transformations in sequence is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

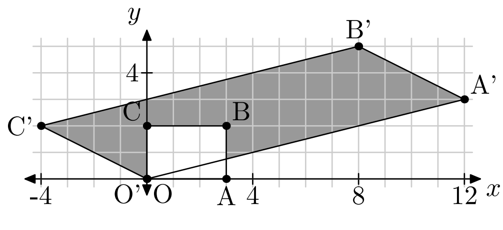
    which is the identity matrix, resulting in the original shape in the original position.

15. (a) $\det \mathrm{T} = 4 \times 1 - (-2) \times 1 = 6$. Given that the area of OABC is 6 units^2, the area of $\mathrm{O'A'B'C'}$ is $6 \times 6 = 36$ units^2.

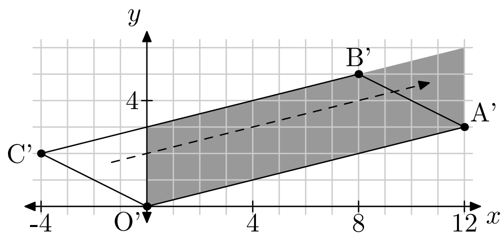
    (b) $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 12 & 8 & -4 \\ 0 & 3 & 5 & 2 \end{bmatrix}$

    The coordinates of O′, A′, B′ and C′ are $(0, 0)$, $(12, 3)$, $(8, 5)$ and $(-4, 2)$ respectively.

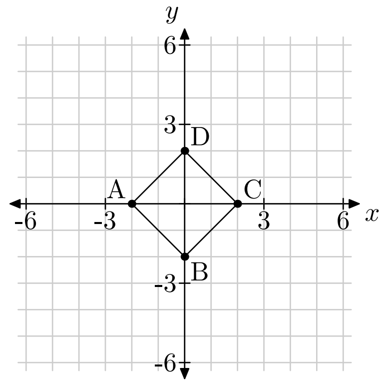
    (c)



    (d) There are number of straightforward ways of determining the area of the parallelogram. For example if we slice off the part of the parallelogram that is left of the y-axis and slide it to the other end (as shown below), we get a parallelogram with a (vertical) base of 3 and (horizontal) perpendicular height of 12, yielding an area of 36.



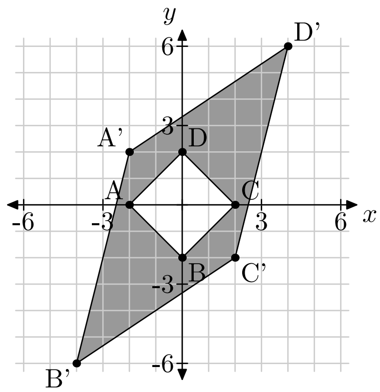
16. (a)



    (b) Area = 8 units^2 (area of any square, rhombus or kite is half the product of its diagonals).

    (c) $\det \mathrm{M} = 1 \times 3 - 2 \times -1 = 5$. The area of $\mathrm{A'B'C'D'}$ is $5 \times 8 = 40$ units^2.

    (d) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 2 & 4 \\ 2 & -6 & -2 & 6 \end{bmatrix}$



    Area = 40 units^2.

17. Every point on the line $y = 2x + 3$ can be represented by $\begin{bmatrix} x \\ 2x+3 \end{bmatrix}$.

    To prove:

$$\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ 2x+3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

    for all x.

    Proof:

$$\begin{aligned} \text{LHS} &= \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ 2x+3 \end{bmatrix} \\ &= \begin{bmatrix} 2(x) - (2x+3) \\ -2(x) + (2x+3) \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 3 \end{bmatrix} \\ &= \text{RHS} \end{aligned}$$

□

    Notice that the matrix $\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$ is singular (i.e. it has a determinant of zero) and therefore is not invertable. This is a requirement of any matrix that transforms two or more distinct points to the same position in the image.

18. Every point on the line $y = x - 1$ can be represented by $\begin{bmatrix} x \\ x-1 \end{bmatrix}$.

$$\begin{aligned}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ x-1 \end{bmatrix} &= \begin{bmatrix} x \\ 2(x)+(x-1) \end{bmatrix}\\ &= \begin{bmatrix} x \\ 3x-1 \end{bmatrix}\end{aligned}$$

The equation of the image line is $y = 3x - 1$.

19. To prove:

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ 3x \end{bmatrix}$$

for all a, b and for some relationship between x and a and b.

Proof:

$$\begin{aligned}\text{LHS} &= \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}\\ &= \begin{bmatrix} a+3b \\ 3a+9b \end{bmatrix}\\ &= \begin{bmatrix} a+3b \\ 3(a+3b) \end{bmatrix}\\ \text{Let } x &= a+3b\\ \text{then LHS} &= \begin{bmatrix} x \\ 3x \end{bmatrix}\\ &= \text{RHS}\end{aligned}$$

□

20. (a) $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ 5-3x \end{bmatrix} = \begin{bmatrix} 6(x)+2(5-3x) \\ 3(x)+(5-3x) \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

The line $y = 5 - 3x$ is transformed to the point $(10, 5)$.

(b) $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6a+2b \\ 3a+b \end{bmatrix}$

Let $x = 6a + 2b$

$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ \frac{x}{2} \end{bmatrix}$

Points on the x-y plane are transformed to the line $y = \frac{x}{2}$ or $2y = x$.

21. Let (a, b) be an arbitrary point before transformation and (a', b') the corresponding point after transformation.

$$\begin{aligned}\begin{bmatrix} a' \\ b' \end{bmatrix} &= \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}\\ &= \begin{bmatrix} 3a \\ 2a+b \end{bmatrix}\end{aligned}$$

If the point before transformation lies on the line $y = m_1x + p$ then $b = m_1a + p$ and the transformed point is

$$\begin{aligned}\begin{bmatrix} a' \\ b' \end{bmatrix} &= \begin{bmatrix} 3a \\ 2a+(m_1a+p) \end{bmatrix}\\ &= \begin{bmatrix} 3a \\ (m_1+2)a+p \end{bmatrix}\end{aligned}$$

We can turn this into a pair of parametric equations then convert that to a Cartesian equation of a line:

$$\begin{aligned}x &= 3a\\ y &= (m_1+2)a+p\\ &= \frac{(m_1+2)(3a)}{3}+p\\ &= \frac{m_1+2}{3}x+p\end{aligned}$$

which is in the form $y = m_2x + p$ where $m_2 = \frac{m_1+2}{3}$, as required.

Now consider two lines perpendicular to each other both before and after transformation.

Let q be the gradient of the first line before transformation.

Since the lines are perpendicular, the gradient of the second line is $-\frac{1}{q}$.

Transforming the first line results in a gradient of $\frac{q+2}{3}$.

Transforming the second line results in a gradient of $\frac{-\frac{1}{q}+2}{3} = \frac{-1+2q}{3q}$.

Since the lines a perpendicular after transformation,

$$\begin{aligned}\frac{q+2}{3} &= -\frac{3q}{-1+2q}\\ &= \frac{3q}{1-2q}\\ (q+2)(1-2q) &= 9q\\ q-2q^2+2-4q &= 9q\\ -2q^2+2-12q &= 0\\ q^2-1+6q &= 0\\ q^2+6q-1 &= 0\\ (q+3)^2-9-1 &= 0\\ (q+3)^2 &= 10\\ q+3 &= \pm\sqrt{10}\\ q &= -3\pm\sqrt{10}\end{aligned}$$

Hence the gradients of the two lines before transformation are $-3+\sqrt{10}$ and $-3-\sqrt{10}$.

Exercise 4C

1. (a) $\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

(b) $\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

$= \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

(e) Two consecutive 30° anticlockwise rotations about the origin are represented by

$$\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 3-1 & -2\sqrt{3} \\ 2\sqrt{3} & -1+3 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

which is a 60° anticlockwise rotation about the origin.

(f) A 30° anticlockwise rotation about the origin followed by a 60° anticlockwise rotation about the origin is represented by

$$\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} \sqrt{3}-\sqrt{3} & -1-3 \\ 3+1 & -\sqrt{3}+\sqrt{3} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is a 90° anticlockwise rotation about the origin.

(g) Two consecutive 45° anticlockwise rotations about the origin are represented by

$$\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= \frac{2}{4}\begin{bmatrix} 1-1 & -1-1 \\ 1+1 & -1+1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is a 90° anticlockwise rotation about the origin.

2. (a) $\begin{bmatrix} \cos(2\times 30) & \sin(2\times 30) \\ \sin(2\times 30) & -\cos(2\times 30) \end{bmatrix}$

$= \begin{bmatrix} \cos 60 & \sin 60 \\ \sin 60 & -\cos 60 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

(b) $\begin{bmatrix} \cos(2\times 60) & \sin(2\times 60) \\ \sin(2\times 60) & -\cos(2\times 60) \end{bmatrix}$

$= \begin{bmatrix} \cos 120 & \sin 120 \\ \sin 120 & -\cos 120 \end{bmatrix}$

$= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

For (a),

$$\left(\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}\right)^2 = \frac{1}{4}\begin{bmatrix} 1+3 & \sqrt{3}-\sqrt{3} \\ \sqrt{3}-\sqrt{3} & 3+1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly for (b),

$$\left(\frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\right)^2 = \frac{1}{4}\begin{bmatrix} 1+3 & -\sqrt{3}+\sqrt{3} \\ -\sqrt{3}+\sqrt{3} & 3+1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Any reflection must be its own inverse since reflecting a reflection restores the original. Consider the general form for a reflection:

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}^2$$
$$= \begin{bmatrix} \cos^2 2\theta + \sin^2 2\theta & \cos 2\theta \sin 2\theta - \cos 2\theta \sin 2\theta \\ \cos 2\theta \sin 2\theta - \cos 2\theta \sin 2\theta & \sin^2 2\theta + \cos^2 2\theta \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. $\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

4. A rotation of angle A followed by a rotation of angle B is equivalent to a rotation of angle $A+B$.

A rotation of angle A followed by a rotation of angle B is represented by $\begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}\begin{bmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{bmatrix}$

$$= \begin{bmatrix} \cos A\cos B - \sin A\sin B & -\cos A\sin B - \sin A\cos B \\ \sin A\cos B + \cos A\sin B & -\sin A\sin B + \cos A\cos B \end{bmatrix}$$
$$= \begin{bmatrix} \cos A\cos B - \sin A\sin B & -(\sin A\cos B + \cos A\sin B) \\ \sin A\cos B + \cos A\sin B & \cos A\cos B - \sin A\sin B \end{bmatrix}$$

A single rotation of angle $A+B$ is represented by $\begin{bmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{bmatrix}$

Equating these gives

$$\begin{bmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{bmatrix}$$
$$= \begin{bmatrix} \cos A\cos B - \sin A\sin B & -(\sin A\cos B + \cos A\sin B) \\ \sin A\cos B + \cos A\sin B & \cos A\cos B - \sin A\sin B \end{bmatrix}$$

Equating corresponding matrix elements from any column or row gives:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\text{and } \cos(A+B) = \cos A\cos B - \sin A\sin B$$

as required. □

5. (a) The 180° rotation is represented by

$$\begin{bmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This transformation leaves point O unchanged at the origin. We need to transform this point to $(6, 4)$ so the total transformation is represented by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

(b) Let θ be the angle that the line OO′ makes with the x-axis. This is the angle that O′A′B′C′ must be rotated clockwise in order to transform O′ onto the x-axis. Since O′ has coordinates $(6, 4)$,

$$\tan\theta = \frac{4}{6} = \frac{2}{3}$$
$$\sin\theta = \frac{4}{\sqrt{4^2+6^2}} = \frac{4}{\sqrt{52}} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}$$
$$\cos\theta = \frac{6}{2\sqrt{13}} = \frac{3}{\sqrt{13}}$$

The matrix to achieve this clockwise rotation (see question 3) is

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$

(c) $\frac{1}{\sqrt{13}}\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 6 & 5 & 5 & 6 \\ 4 & 4 & 3 & 3 \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 26 & 23 & 21 & 24 \\ 0 & 2 & -1 & -3 \end{bmatrix}$

- O″$(\frac{26}{\sqrt{13}}, 0) = (2\sqrt{13}, 0)$,
- A″$(\frac{23}{\sqrt{13}}, \frac{2}{\sqrt{13}})$,
- B″$(\frac{21}{\sqrt{13}}, -\frac{1}{\sqrt{13}})$,
- C″$(\frac{24}{\sqrt{13}}, -\frac{3}{\sqrt{13}})$.

(You could, if preferred, give these with rational denominators and arrive at the same answers Sadler gives.)

Miscellaneous Exercise 4

1. Choose every possibility where the number of columns in the first is equal to the number of rows in the second. Thus

   - A has 3 columns so it can pre-multiply every matrix with 3 rows, resulting in the products AC and AD.
   - B also has 3 columns, resulting in products BC and BD.
   - C has only 1 column so it can pre-multiply every matrix with one row: CB.
   - D has three columns, resulting in the products DC and D^2.

2. No working necessary for these questions.

3. XY cannot be formed – X has three columns but Y has only one row.

   YX cannot be formed – Y has three columns but X has five rows.

   XZ can be formed – X has three columns and Z has three rows.

   ZX cannot be formed – Z has one column and X has five rows.

   The product XZ has five rows (number of rows in X) and one column (number of columns in Z). Each row is the sum of the number of wins times the number of points per win, the number of draws times the number of points per draw and the number of losses times the number of points per loss, that is, the total points for the corresponding team.

4. (a)
$$\begin{aligned} 4e^{\pi i/6} &= 4\operatorname{cis}\frac{\pi}{6} \\ &= 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\ &= 4\times\frac{\sqrt{3}}{2} + 4\times\frac{1}{2}i \\ &= 2\sqrt{3} + 2i \end{aligned}$$

(b) $$\begin{aligned} -20e^{\pi i/3} &= -20\,\mathrm{cis}\,\frac{\pi}{3} \\ &= -20\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ &= -20\times\frac{1}{2} - 20\times\frac{\sqrt{3}}{2}i \\ &= -10 - 10\sqrt{3}i \end{aligned}$$

(c) $$\begin{aligned} 20e^{-\pi i/3} &= 20\,\mathrm{cis}\left(-\frac{\pi}{3}\right) \\ &= 20\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) \\ &= 20\times\frac{1}{2} - 20\times\frac{\sqrt{3}}{2}i \\ &= 10 - 10\sqrt{3}i \end{aligned}$$

(d) $$\begin{aligned} 1 + e^{\pi i/2} &= 1 + \mathrm{cis}\,\frac{\pi}{2} \\ &= 1 + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= 1 + 0 + i \\ &= 1 + i \end{aligned}$$

5. $$\begin{aligned} P &= Q + PR \\ P - PR &= Q \\ P(I - R) &= Q \\ P &= Q(I-R)^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -7 \\ 5 & -8 \end{bmatrix} \end{aligned}$$

6. $$\begin{aligned} AB &= \begin{bmatrix} x & 2 \\ y & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3x-2 & x+8 \\ 3y-1 & y+4 \end{bmatrix} \\ BA &= \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x & 2 \\ y & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3x+y & 6+1 \\ -x+4y & -2+4 \end{bmatrix} \\ &= \begin{bmatrix} 3x+y & 7 \\ -x+4y & 2 \end{bmatrix} \end{aligned}$$

Given AB=BA, we can equate corresponding matrix elements. From element 1,2:

$$\begin{aligned} x + 8 &= 7 \\ x &= -1 \end{aligned}$$

From element 2,2:

$$\begin{aligned} y + 4 &= 2 \\ y &= -2 \end{aligned}$$

Confirm these results by substitution:

$$\begin{aligned} AB &= \begin{bmatrix} 3(-1)-2 & 7 \\ 3(-2)-1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 7 \\ -7 & 2 \end{bmatrix} \\ BA &= \begin{bmatrix} 3(-1)+(-2) & 7 \\ -(-1)+4(-2) & 2 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 7 \\ -7 & 2 \end{bmatrix} \end{aligned}$$

Hence

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ -7 & 2 \end{bmatrix}$$

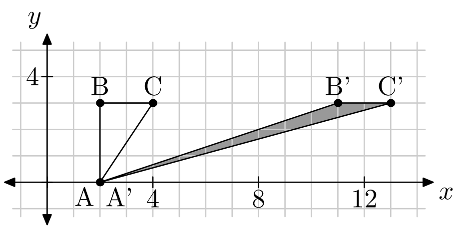
$$\begin{aligned} \therefore \quad p &= -5 \\ q &= 7 \\ r &= -7 \\ s &= 2 \end{aligned}$$

7. $$\begin{aligned} A\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix} \\ A &= \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}\frac{1}{4-3}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \end{aligned}$$

8. $$\begin{aligned} k &= z_1^4 \\ &= (2\,\mathrm{cis}\,40°)^4 \\ &= 2^4\,\mathrm{cis}(40\times 4)° \\ &= 16\,\mathrm{cis}\,160°. \end{aligned}$$

The other roots are equal to z_1 rotated a multiple of 90°, i.e. $2\,\mathrm{cis}(130°)$, $2\,\mathrm{cis}(-50°)$ and $2\,\mathrm{cis}(-140°)$.

9. $$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 11 & 13 \\ 0 & 3 & 3 \end{bmatrix}$$



This represents a shear parallel to the x-axis with scale factor 3.

10. To prove:

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n}{4}(n+1)(n+2)(n+3)$$

Assume this is true for some $n = k$, then for $n = k+1$:

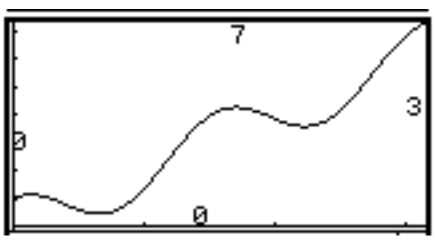
$$\begin{aligned}
\text{LHS} &= \sum_{i=1}^{k+1} i(i+1)(i+2) \\
&= \left(\sum_{i=1}^{k} i(i+1)(i+2)\right) + (k+1)(k+2)(k+3) \\
&= \frac{k}{4}(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3) \\
&= (k+1)(k+2)(k+3)(\frac{k}{4}+1) \\
&= (k+1)(k+2)(k+3)(\frac{k+4}{4}) \\
&= \frac{1}{4}(k+1)(k+2)(k+3)(k+4) \\
&= \frac{k+1}{4}(k+2)(k+3)(k+4) \\
&= \text{RHS}
\end{aligned}$$

Therefore, if it is true for some $n = k$ then it is also true for $n = k+1$.

For $n = 1$,

$$\begin{aligned}
\text{LHS} &= \sum_{i=1}^{1} i(i+1)(i+2) \\
&= 1(2)(3) \\
&= (1)(2)(3)\frac{4}{4} \\
&= \frac{1}{4}(2)(3)(4) \\
&= \text{RHS}
\end{aligned}$$

Therefore the proposition is true for $n = 1$ and hence by mathematical induction, true for all $n, n \geq 1$. □

11. 

(a) From the graph, the global maximum appears to be where $x = \pi$ with coordinates $(\pi, 2\pi + \cos 4\pi) = (\pi, 2\pi + 1)$. (This is not actually a local maximum, since the gradient at that point is positive.)

(b) From the graph, the global minimum corresponds to the first local minimum which is the second stationary point.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 4\sin 4x$$

at the stationary points,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
2 - 4\sin 4x &= 0 \\
\sin 4x &= \frac{1}{2} \\
4x &= \frac{5\pi}{6}
\end{aligned}$$

(ignoring the first solution at $4x = \frac{\pi}{6}$ as this will yield the first stationary point and we want the second)

$$\begin{aligned}
x &= \frac{5\pi}{24} \\
y &= \frac{5\pi}{12} + \cos\frac{5\pi}{6} \\
&= \frac{5\pi}{12} - \frac{\sqrt{3}}{2} \\
&= \frac{5\pi - 6\sqrt{3}}{12}
\end{aligned}$$

Thus the coordinates of the global minimum are $(\frac{5\pi}{24}, \frac{5\pi-6\sqrt{3}}{12})$.

(c) The local minimum that is not the global minimum is the fourth stationary point:

$$\begin{aligned}
\sin 4x &= \frac{1}{2} \\
4x &= \frac{17\pi}{6} \\
x &= \frac{17\pi}{24} \\
y &= \frac{17\pi}{12} + \cos\frac{5\pi}{6} \\
&= \frac{17\pi}{12} - \frac{\sqrt{3}}{2} \\
&= \frac{17\pi - 6\sqrt{3}}{12}
\end{aligned}$$

Thus the coordinates of the local minimum are $(\frac{17\pi}{24}, \frac{17\pi-6\sqrt{3}}{12})$.

(d) The two local minima are the first and third stationary points

$$\sin 4x = \frac{1}{2}$$

$$\begin{aligned}
4x &= \frac{\pi}{6} & &\text{or} & 4x &= \frac{13\pi}{6} \\
x &= \frac{\pi}{24} & &\text{or} & x &= \frac{13\pi}{24} \\
y &= \frac{\pi}{12} + \cos\frac{\pi}{6} & &\text{or} & y &= \frac{13\pi}{12} + \cos\frac{\pi}{6} \\
&= \frac{\pi}{12} + \frac{\sqrt{3}}{2} & & & &= \frac{13\pi}{12} + \frac{\sqrt{3}}{2} \\
&= \frac{\pi + 6\sqrt{3}}{12} & & & &= \frac{13\pi + 6\sqrt{3}}{12}
\end{aligned}$$

Thus the coordinates of the local minimum are $(\frac{\pi}{24}, \frac{\pi+6\sqrt{3}}{12})$ and $(\frac{13\pi}{24}, \frac{13\pi+6\sqrt{3}}{12})$.

12.
$$\begin{aligned}
\operatorname{cis} 4\theta &= (\operatorname{cis}\theta)^4 \\
&= (\cos\theta + \mathrm{i}\sin\theta)^4 \\
&= \cos^4\theta \\
&\quad + 4\cos^3\theta(\mathrm{i}\sin\theta) \\
&\quad + 6\cos^2\theta(\mathrm{i}^2\sin^2\theta) \\
&\quad + 4\cos\theta(\mathrm{i}^3\sin^3\theta) \\
&\quad + \mathrm{i}^4\sin^4\theta \\
&= \cos^4\theta \\
&\quad + 4\mathrm{i}\cos^3\theta\sin\theta \\
&\quad - 6\cos^2\theta\sin^2\theta \\
&\quad - 4\mathrm{i}\cos\theta\sin^3\theta \\
&\quad + \sin^4\theta \\
&= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) \\
&\quad + \mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta) \\
\operatorname{Re}(\operatorname{cis}4\theta) &= \cos 4\theta \\
&= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\
\operatorname{Im}(\operatorname{cis}4\theta) &= \sin 4\theta \\
&= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta
\end{aligned}$$

13.
$$\begin{aligned}
\mathrm{AP} + \mathrm{BP} + \mathrm{P} &= \mathrm{Q} \\
\mathrm{AP} + \mathrm{BP} + \mathrm{IP} &= \mathrm{Q} \\
(\mathrm{A} + \mathrm{B} + \mathrm{I})\mathrm{P} &= \mathrm{Q} \\
\mathrm{P} &= (\mathrm{A} + \mathrm{B} + \mathrm{I})^{-1}\mathrm{Q} \\
\mathrm{A} + \mathrm{B} + \mathrm{I} &= \begin{bmatrix} 3 & -1 \\ 8 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 8 & 2 \\ 7 & 2 \end{bmatrix} \\
\mathrm{P} &= \begin{bmatrix} 8 & 2 \\ 7 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -7 & 8 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} -2 & 2 \\ 6 & -10 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \\ 3 & -5 \end{bmatrix}
\end{aligned}$$

14. (a) To prove:

$$\mathrm{AB}^{-1} = \mathrm{B}^{-1}\mathrm{A}$$

Proof:

$$\begin{aligned}
\mathrm{LHS} &= \mathrm{AB}^{-1} \\
&= \mathrm{IAB}^{-1} \\
&= \mathrm{B}^{-1}\mathrm{BAB}^{-1} \\
&= \mathrm{B}^{-1}\mathrm{ABB}^{-1} \\
&= \mathrm{B}^{-1}\mathrm{AI} \\
&= \mathrm{B}^{-1}\mathrm{A} \\
&= \mathrm{RHS}
\end{aligned}$$

□

(b) To prove:

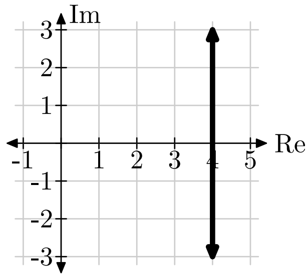
$$\mathrm{BA}^{-1} = \mathrm{A}^{-1}\mathrm{B}$$

Proof:

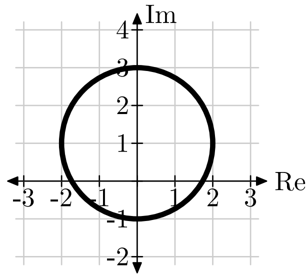
$$\begin{aligned}
\mathrm{LHS} &= \mathrm{BA}^{-1} \\
&= \mathrm{IBA}^{-1} \\
&= \mathrm{A}^{-1}\mathrm{ABA}^{-1} \\
&= \mathrm{A}^{-1}\mathrm{BAA}^{-1} \\
&= \mathrm{A}^{-1}\mathrm{BI} \\
&= \mathrm{A}^{-1}\mathrm{B} \\
&= \mathrm{RHS}
\end{aligned}$$

□

15. (a)
$$\begin{aligned}
z + \bar{z} &= 2\operatorname{Re}(z) \\
\text{so } z + \bar{z} &= 4 \\
\text{becomes } 2\operatorname{Re}(z) &= 4 \\
\operatorname{Re}(z) &= 2
\end{aligned}$$



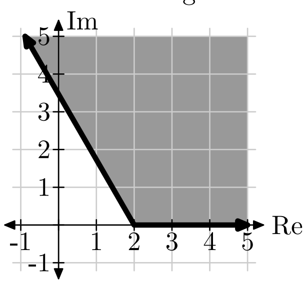
(b) This is a circle centred at $0 + \mathrm{i}$ and radius 2:



(c) This is a region the same shape as

$$0 \le \arg(z) \le \frac{2\pi}{3}$$

but translated 2 units right:



16. There is no conflict. The "proof" supposes that A^{-1} exists. In neither of the examples is this the case. In Example 1, A is not a square matrix, and hence no inverse exists. In Example 2, A is square, but it is singular so again no inverse exists.

What the proof actually shows is that if AB = AC **and** A **is a non-singular square matrix**, then B = C.

17. Let $\theta = \angle AOB$.

Let s be the slant height of the resulting cone. This is equal to the radius of the original circle, that is, $s = AO$.

Let r be the radius of the cone.

Let h be the perpendicular height of the cone.

The circumference of the original circle is $2\pi s$. The fraction of this that is removed by sector AOB is $\frac{\theta}{2\pi}$, so the fraction remaining is

$$1 - \frac{\theta}{2\pi} = \frac{2\pi - \theta}{2\pi}$$

with the length of the remaining part of the circumference given by

$$l = \frac{2\pi - \theta}{2\pi}(2\pi s)$$

This becomes the circumference of the base of the cone, giving us the radius of the cone

$$\begin{aligned} r &= \frac{l}{2\pi} \\ &= \frac{(2\pi - \theta)s}{2\pi} \end{aligned}$$

The perpendicular height of the cone can be found using Pythagoras' theorem since the slant height, perpendicular height and radius of the cone form a right triangle:

$$\begin{aligned} h &= \sqrt{s^2 - r^2} \\ &= \sqrt{s^2 - \frac{(2\pi - \theta)^2 s^2}{4\pi^2}} \\ &= \sqrt{\frac{s^2}{4\pi^2}\left(4\pi^2 - (2\pi - \theta)^2\right)} \\ &= \frac{s}{2\pi}\sqrt{4\pi^2 - (2\pi - \theta)^2} \\ &= \frac{s}{2\pi}\sqrt{4\pi^2 - (4\pi^2 - 4\pi\theta + \theta^2)} \\ &= \frac{s}{2\pi}\sqrt{4\pi\theta - \theta^2} \end{aligned}$$

The volume of the cone is given by

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\left(\frac{(2\pi - \theta)s}{2\pi}\right)^2 \frac{s}{2\pi}\sqrt{4\pi\theta - \theta^2} \\ &= \frac{(1)\left((2\pi - \theta)s\right)^2 (s)\sqrt{4\pi\theta - \theta^2}}{(3)(2\pi)^2(2\pi)} \\ &= \frac{(2\pi - \theta)^2 s^2 (s)\sqrt{4\pi\theta - \theta^2}}{3(2\pi)^3} \\ &= \frac{(2\pi - \theta)^2 s^3 \sqrt{4\pi\theta - \theta^2}}{3(2\pi)^3} \\ &= \frac{s^3}{3(2\pi)^3}(2\pi - \theta)^2\sqrt{4\pi\theta - \theta^2} \end{aligned}$$

Differentiating with respect to θ (and bearing in mind that s is constant):

$$\begin{aligned} \frac{dV}{d\theta} &= \frac{s^3}{3(2\pi)^3}\left(2(2\pi - \theta)(-1)\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^2(4\pi - 2\theta)}{2\sqrt{4\pi\theta - \theta^2}}\right) \\ &= \frac{s^3}{3(2\pi)^3}\left(-2(2\pi - \theta)\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^2 2(2\pi - \theta)}{2\sqrt{4\pi\theta - \theta^2}}\right) \\ &= \frac{s^3}{3(2\pi)^3}\left(-2(2\pi - \theta)\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^3}{\sqrt{4\pi\theta - \theta^2}}\right) \\ &= \frac{s^3}{3(2\pi)^3}\,\frac{-2(2\pi - \theta)(4\pi\theta - \theta^2) + (2\pi - \theta)^3}{\sqrt{4\pi\theta - \theta^2}} \end{aligned}$$

At the maximum volume, this derivative is zero.

$$\begin{aligned} \frac{s^3}{3(2\pi)^3}\,\frac{-2(2\pi - \theta)(4\pi\theta - \theta^2) + (2\pi - \theta)^3}{\sqrt{4\pi\theta - \theta^2}} &= 0 \\ -2(2\pi - \theta)(4\pi\theta - \theta^2) + (2\pi - \theta)^3 &= 0 \\ -2(4\pi\theta - \theta^2) + (2\pi - \theta)^2 &= 0 \end{aligned}$$

(the previous step is only valid because we know $2\pi - \theta \neq 0$)

$$\begin{aligned} -8\pi\theta + 2\theta^2 + 4\pi^2 - 4\pi\theta + \theta^2 &= 0 \\ 3\theta^2 - 12\pi\theta + 4\pi^2 &= 0 \\ 3(\theta - 2\pi)^2 - 12\pi^2 + 4\pi^2 &= 0 \\ 3(\theta - 2\pi)^2 - 8\pi^2 &= 0 \\ 3(\theta - 2\pi)^2 &= 8\pi^2 \\ (\theta - 2\pi)^2 &= \frac{2}{3}(2\pi)^2 \\ \theta - 2\pi &= \pm\sqrt{\frac{2}{3}}(2\pi) \\ \theta &= 2\pi \pm \frac{\sqrt{6}}{3}(2\pi) \\ &= 2\pi\left(1 - \frac{\sqrt{6}}{3}\right) \\ &= \frac{2\pi(3 - \sqrt{6})}{3} \end{aligned}$$

(in the second last step, discarding the result that would result in $\theta > 2\pi$.)

Converting to degrees gives

$$\begin{aligned} \theta &= 120(3 - \sqrt{6}) \\ &= 66.1^\circ \text{ (1 d.p.)} \end{aligned}$$