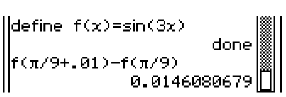
Chapter 7

Exercise 7A

1.
$$\begin{aligned} f'(x) &= 3x^2 - 5 \\ f'(5) &= 3(5)^2 - 5 \\ &= 70 \\ \frac{\delta f(x)}{\delta x} &\approx f'(x) \\ \delta f(5) &\approx f'(5)\delta x \\ &= 70 \times 0.01 \\ &= 0.7 \\ f(5.01) - f(5) &= (5.01)^3 - 5(5.01) - (5)^3 + 5(5) \\ &= 0.701501 \end{aligned}$$

2.
$$\begin{aligned} f'(x) &= 10x + 2 \\ f'(10) &= 10(10) + 2 \\ &= 102 \\ \frac{\delta f(x)}{\delta x} &\approx f'(x) \\ \delta f(10) &\approx f'(10)\delta x \\ &= 102 \times 0.1 \\ &= 10.2 \\ f(10.1) - f(10) &= 5(10.1)^2 + 2(10.1) - 5(10)^2 \\ &\quad - 2(10) \\ &= 10.25 \end{aligned}$$

3.
$$\begin{aligned} f'(x) &= 3\cos 3x \\ f'(\frac{\pi}{9}) &= 3\cos\frac{\pi}{3} \\ &= 1.5 \\ \frac{\delta f(x)}{\delta x} &\approx f'(x) \\ \delta f(\frac{\pi}{9}) &\approx f'(\frac{\pi}{9})\delta x \\ &= 1.5 \times 0.01 \\ &= 0.015 \\ f(\frac{\pi}{9} + 0.01) - f(\frac{\pi}{9}) &= 0.0146 \end{aligned}$$



4.
$$\begin{aligned} f'(x) &= 30\sin^2 5x\cos 5x \\ f'(\frac{\pi}{5}) &= 30\sin^2\frac{5\pi}{3}\cos\frac{5\pi}{3} \\ &= 30\left(-\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right) \\ &= 15\left(\frac{3}{4}\right) \\ &= 11.25 \\ \delta f(\frac{\pi}{3}) &\approx f'(\frac{\pi}{3})\delta x \\ &= 11.25 \times 0.001 \\ &= 0.01125 \\ f(\frac{\pi}{3} + 0.001) - f(\frac{\pi}{3}) &= 0.011266 \end{aligned}$$

5. The marginal cost is $\frac{\mathrm{d}}{\mathrm{d}x}C(x)$. As the expressions given can be easily differentiated, no working is required.

6. Marginal cost $C'(x) = \frac{\mathrm{d}C}{\mathrm{d}x} = \frac{10}{\sqrt{x}}$

   (a) $C'(25) = \frac{10}{\sqrt{25}}$
   $= \$2$ per unit

   (b) $C'(100) = \frac{10}{\sqrt{100}}$
   $= \$1$ per unit

   (c) $C'(400) = \frac{10}{\sqrt{400}}$
   $= \$0.50$ per unit

7. Marginal cost is $C'(x) = 750 - 30x + 0.3x^2$ dollars per tonne.

   (a) $C'(30) = 750 - 30(30) + 0.3(30)^2$
   $= \$120$ per tonne

   (b) $C'(60) = 750 - 30(60) + 0.3(60)^2$
   $= \$30$ per tonne

   (c) $C'(100) = 750 - 30(100) + 0.3(100)^2$
   $= \$750$ per tonne

8. $C'(x) = x$
   $C'(10) = \$10$ per item

   It costs approximately an additional \$10 to produce the next item. (It's only approximate because $\frac{\delta C}{\delta x} \approx \frac{\mathrm{d}C}{\mathrm{d}x}$.)

9. (a) $P(2) = R(2) - C(2)$
   $= 850 - 446$
   $= \$404$

   (b) $\frac{P(2)}{2} = \$202$ profit per item

(c) $$\begin{aligned} P(10) &= R(10) - C(10) \\ &= 4250 - 1150 \\ &= \$3100 \end{aligned}$$

(d) $\frac{P(10)}{10} = \$310$ profit per item

(e) $$\begin{aligned} P'(x) &= R'(x) - C'(x) \\ &= 425 - (200 - 60x + 6x^2) \\ &= 225 + 60x - 6x^2 \end{aligned}$$

At maximum profit,

$$\begin{aligned} P'(x) &= 0 \\ 225 + 60x - 6x^2 &= 0 \\ x &= 12.9 \end{aligned}$$

(ignoring the negative root). Thus, to the nearest integer, the maximum profit will be achieved when 13 items are produced. This profit is

$$\begin{aligned} P(13) &= R(13) - C(13 \\ &= 5525 - 2074 \\ &= \$3451 \end{aligned}$$

10. (a) $$\begin{aligned} C(x) &= \int (4x+2)\,\mathrm{d}x \\ &= 2x^2 + 2x + c \\ C(0) &= 100 \\ c &= 100 \\ C(x) &= 2x^2 + 2x + 100 \end{aligned}$$

(b) $$\begin{aligned} C(x) &= \int x(3x+4)\,\mathrm{d}x \\ &= \int (3x^2 + 4x)\,\mathrm{d}x \\ &= x^3 + 2x^2 + c \\ C(0) &= 8000 \\ x &= 8000 \\ C(x) &= x^3 + 2x^2 + 8000 \end{aligned}$$

(c) $$\begin{aligned} C(x) &= \int 20(5x + \sin 2x)\,\mathrm{d}x \\ &= 20\left(\frac{5x^2}{2} - \frac{\cos 2x}{2}\right) + c \\ &= 10\left(5x^2 - \cos 2x\right) + c \\ C(0) &= 30 \\ 10\left(-\cos 0\right) + c &= 30 \\ -10 + c &= 30 \\ c &= 40 \\ C(x) &= 10\left(5x^2 - \cos 2x\right) + 40 \\ &= 10\left(5x^2 - \cos 2x + 4\right) \end{aligned}$$

(d) $$\begin{aligned} C(x) &= \int 4(3x^3 + 5\cos^2 x)\,\mathrm{d}x \\ &= 3x^4 + 10\int (2\cos^2 x - 1 + 1)\,\mathrm{d}x \\ &= 3x^4 + 10\int (\cos 2x + 1)\,\mathrm{d}x \\ &= 3x^4 + 5\sin 2x + 10x + c \\ C(0) &= 100 \\ c &= 100 \\ C(x) &= 3x^4 + 5\sin 2x + 10x + 100 \end{aligned}$$

11. For a revenue function, it is usually safe to assume $R(0) = 0$.

(a) $$\begin{aligned} R(x) &= \int 500\,\mathrm{d}x \\ &= 500x \end{aligned}$$

(b) $$\begin{aligned} R(x) &= \int (60 - 0.1x)\,\mathrm{d}x \\ &= 60x - 0.05x^2 \end{aligned}$$

12. (a) $$\begin{aligned} N(x) &= \int 40\pi \cos\frac{\pi t}{600} dt \\ &= 24\,000 \sin\frac{\pi t}{600} + c \\ N(0) &= 0 \\ c &= 0 \\ N(x) &= 24\,000 \sin\frac{\pi t}{600} \end{aligned}$$

(b) $$\begin{aligned} N(100) &= 24\,000 \sin\frac{100\pi}{600} \\ &= 24\,000 \sin\frac{\pi}{6} \\ &= 12\,000 \end{aligned}$$

(c) $$\begin{aligned} N(100) - N(99) &= 12\,000 - 24\,000 \sin\frac{99\pi}{600} \\ &= 12\,000 - 11\,891 \\ &= 109 \end{aligned}$$

Alternatively estimate using

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= 40\pi \cos\frac{100\pi}{600} \\ &= 40\pi \cos\frac{\pi}{6} \\ &= 20\pi\sqrt{3} \\ &= 108.8 \\ &\approx 109 \end{aligned}$$

13. $$\begin{aligned} C(x) &= \int \frac{20}{\sqrt{x}}\,\mathrm{d}x \\ &= 40\sqrt{x} + c \\ C(0) &= 500 \\ c &= 500 \\ C(x) &= 40\sqrt{x} + 500 \\ C(100) &= 40\sqrt{100} + 500 \\ &= \$900 \\ C(400) &= 40\sqrt{400} + 500 \\ &= \$1\,300 \end{aligned}$$

    Average cost

    (a) for $x = 100$ is $\frac{900}{100} = \$9$ per unit
    (b) for $x = 400$ is $\frac{1300}{400} = \$3.25$ per unit

14. $$\begin{aligned} &\int_{25}^{30} (x^2 + 10x)\,\mathrm{d}x \\ &= \left[\frac{x^3}{3} + 5x^2\right]_{25}^{30} \\ &= (9000 + 4500) - (5208\frac{1}{3} + 3125) \\ &= 13\,500 - 8\,333\frac{1}{3} \\ &= 5166\frac{2}{3} \\ &\approx \$5200 \end{aligned}$$

15. (a) Note that $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$ for $t < 4$ so

    $$\int_0^1 \frac{\mathrm{d}V}{\mathrm{d}t} dt$$

    gives the opposite of the amount drained, so the integral we need is

    $$\begin{aligned} -\int_0^1 \frac{\mathrm{d}V}{\mathrm{d}t} dt &= -\int_0^1 (5t - 40) dt \\ &= \int_0^1 (40 - 5t) dt \end{aligned}$$

    (b) $$\begin{aligned} \int_3^4 (40 - 5t) dt &= \left[40t - \frac{5t^2}{2}\right]_3^4 \\ &= (160 - 40) - (120 - 22.5) \\ &= 22.5 \text{ kL} \end{aligned}$$

    (c) $$\begin{aligned} \int_0^4 (40 - 5t) dt &= \left[40t - \frac{5t^2}{2}\right]_0^4 \\ &= (160 - 40) - 0 \\ &= 120 \text{ kL} \end{aligned}$$

16. Let θ be the size in radians of the nominally $60°$ angle. Let h be the height of the triangle. The area of the triangle is

    $$\begin{aligned} A &= \frac{1}{2} 8h \\ &= 4h \\ \tan\theta &= \frac{h}{8} \\ h &= 8\tan\theta \\ A &= 4(8\tan\theta) \\ &= 32\tan\theta \\ \frac{\mathrm{d}A}{\mathrm{d}\theta} &= \frac{32}{\cos^2\theta} \\ \frac{\Delta A}{\Delta\theta} &\approx \frac{32}{\cos^2\theta} \\ \Delta A &\approx \frac{32}{\cos^2\theta}\Delta\theta \\ &= \frac{32}{\cos^2 60°}\left(\frac{\pi \times 0.5}{180}\right) \\ &= 128\left(\frac{\pi}{360}\right) \\ &\approx 1.1\text{cm}^2 \end{aligned}$$

Exercise 7B

1. No working required.

2. (a) Initial displacement is $x = 5(0)^2 - 2(0) + 8 = 8$m

    (b) Initial distance is $|x| = 8$m

    (c) Initial velocity:

    $$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 10t - 2 \\ \frac{\mathrm{d}v}{\mathrm{d}t}_{|t=0} &= -2\text{ms}^{-1} \end{aligned}$$

    (d) Speed at $t = 2$ is $|v|_{|t=2} = 10(2) - 2 = 18\text{ms}^{-1}$

(e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 10\text{ms}^{-2} \end{aligned}$$

3. (a) Initial displacement is $x = 0(2(0)+1) = 0$m
   (b) Initial distance is $|x| = 0$m
   (c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 4t + 1 \\ \frac{\mathrm{d}v}{\mathrm{d}t}_{|t=0} &= 1\text{ms}^{-1} \end{aligned}$$

   (d) Speed at $t = 2$ is $|v|_{|t=2} = 4(2)+1 = 9\text{ms}^{-1}$
   (e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 4\text{ms}^{-2} \end{aligned}$$

4. (a) Initial displacement is $x = 4(0)^4 + 3(0) = 0$m
   (b) Initial distance is $|x| = 0$m
   (c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 12t^2 + 3 \\ \frac{\mathrm{d}v}{\mathrm{d}t}_{|t=0} &= 3\text{ms}^{-1} \end{aligned}$$

   (d) Speed at $t = 2$ is $|v|_{|t=2} = 12(2)^2 + 3 = 51\text{ms}^{-1}$
   (e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 24t \\ a_{|t=5} &= 120\text{ms}^{-2} \end{aligned}$$

5. (a) Initial displacement is $x = 30 - 6(0) = 30$m
   (b) Initial distance is $|x| = 30$m
   (c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -6\text{ms}^{-1} \end{aligned}$$

   (d) Speed at $t = 2$ is $|v|_{|t=2} = |(| - 6) = 6\text{ms}^{-1}$
   (e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 0\text{ms}^{-2} \end{aligned}$$

6. (a) Initial displacement is $x = 2(0)^3 - 30(0) - 1 = -1$m
   (b) Initial distance is $|x| = 1$m
   (c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 6t^2 - 30 \\ \frac{\mathrm{d}v}{\mathrm{d}t}_{|t=0} &= -30\text{ms}^{-1} \end{aligned}$$

   (d) Speed at $t = 2$ is $|v|_{|t=2} = |6(2)^2 - 30| = 6\text{ms}^{-1}$
   (e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 12t \\ a_{|t=5} &= 60\text{ms}^{-2} \end{aligned}$$

7. (a) Initial displacement is $x = (1-4(0))^3 = 1$m
   (b) Initial distance is $|x| = 1$m
   (c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 3(1-4t)^2(-4) \\ &= -12(1-4t)^2 \\ \frac{\mathrm{d}v}{\mathrm{d}t}_{|t=0} &= -12\text{ms}^{-1} \end{aligned}$$

   (d) Speed at $t = 2$ is

$$\begin{aligned} |v|_{|t=2} &= |-12(1-4(2))^2| \\ &= 12 \times 49 \\ &= 588\text{ms}^{-1} \end{aligned}$$

   (e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= -24(1-4t)(-4) \\ &= 96(1-4t) \\ a_{|t=5} &= 96(1-20) \\ &= -1824\text{ms}^{-2} \end{aligned}$$

8. (a)

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 6t + 4 \\ v_{|t=3} &= 6(3) + 4 \\ &= 22\,\text{m/s} \end{aligned}$$

   (b)

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 6\,\text{m/s}^2 \end{aligned}$$

9. (a)

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 12t \\ a_{|t=3} &= 12(3) \\ &= 36\,\text{m/s}^2 \end{aligned}$$

(b)
$$\begin{aligned}
x &= \int v\,\mathrm{d}t \\
&= 2t^3 - t + c \\
51 &= 2(1)^3 - (1) + c \\
c &= 50 \\
x &= 2t^3 - t + 50 \\
x_{t=4} &= 2(4)^3 - 4 + 50 \\
&= 174\,\mathrm{m}
\end{aligned}$$

Another possible approach is to use the known position and a definite integral from the known time:

$$\begin{aligned}
x &= 51 + \int_1^4 (6t^2 - 1)\,\mathrm{d}t \\
&= 51 + \left[2t^3 - t\right]_1^4 \\
&= 51 + \left(2(4)^3 - 4\right) - \left(2(1)^3 - 1\right) \\
&= 51 + 124 - 1 \\
&= 174\,\mathrm{m}
\end{aligned}$$

10. (a)
$$\begin{aligned}
a_{|t=0} &= 6(0) + 4 \\
&= 4\,\mathrm{m/s^2}
\end{aligned}$$

(b)
$$\begin{aligned}
v &= \int (6t + 4)\,\mathrm{d}t \\
&= 3t^2 + 4t + c \\
8 &= 3(1)^2 + 4(1) + c \\
c &= 1 \\
v &= 3t^2 + 4t + 1 \\
v_{t=3} &= 3(3)^2 + 4(3) + 1 \\
&= 40\,\mathrm{m/s}
\end{aligned}$$

(c)
$$\begin{aligned}
x_{t=2} &= 9 + \int_1^2 (3t^2 + 4t + 1)dt \\
&= 9 + \left[t^3 + 2t^2 + t\right]_1^2 \\
&= 9 + \left((2)^3 + 2(2)^2 + 2\right) \\
&\quad - \left((1)^3 + 2(1)^2 + 1\right) \\
&= 9 + 18 - 4 \\
&= 23\,\mathrm{m}
\end{aligned}$$

11. (a)
$$\begin{aligned}
a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\
&= 6\sqrt{16 + t^2} + \frac{6t\left(\frac{1}{2}\right)(2t)}{\sqrt{16 + t^2}} \\
&= 6\sqrt{16 + t^2} + \frac{6t^2}{\sqrt{16 + t^2}} \\
&= \frac{6(16 + t^2) + 6t^2}{\sqrt{16 + t^2}} \\
&= \frac{96 + 6t^2 + 6t^2}{\sqrt{16 + t^2}} \\
&= \frac{96 + 12t^2}{\sqrt{16 + t^2}} \\
&= 12\frac{8 + t^2}{\sqrt{16 + t^2}} \\
a_{|t=0} &= 12\frac{8 + (0)^2}{\sqrt{16 + (0)^2}} \\
&= 24\,\mathrm{m/s^2}
\end{aligned}$$

(b)
$$\begin{aligned}
x &= \int v\,\mathrm{d}t \\
&= \int (6t\sqrt{16 + t^2})\,\mathrm{d}t \\
&= 3\int 2t(16 + t^2)^{0.5}\,\mathrm{d}t \\
&= \frac{3(16 + t^2)^{1.5}}{1.5} + c \\
&= 2(16 + t^2)^{1.5} + c \\
8 &= 2(16 + 0^2)^{1.5} + c \\
8 &= 128 + c \\
c &= -120 \\
x &= 2(16 + t^2)^{1.5} - 120 \\
x_{|t=3} &= 2(16 + (3)^2)^{1.5} - 120 \\
&= 2(25)^{1.5} - 120 \\
&= 2(125) - 120 \\
&= 250 - 120 \\
&= 130\,\mathrm{m}
\end{aligned}$$

12.
$$\begin{aligned}
v &= \int a\,\mathrm{d}t \\
&= \int \frac{6t(t^2 + 2t + 1)}{5}\,\mathrm{d}t \\
&= \int \frac{6t^3 + 12t^2 + 6t}{5}\,\mathrm{d}t \\
&= 0.3t^4 + 0.8t^3 + 0.6t^2 + c \\
2 &= 0.3(1)^4 + 0.8(1)^3 + 0.6(1)^2 + c \\
2 &= 1.7 + c \\
c &= 0.3 \\
v_{|t=0} &= 0.3\,\mathrm{m/s}
\end{aligned}$$

13. (a)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -2\sin t \\ v_{|t=\frac{\pi}{6}} &= -2\sin\frac{\pi}{6} \\ &= -1\,\mathrm{m/s} \end{aligned}$$

(b)
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= -2\cos t \\ a_{|t=\frac{\pi}{2}} &= -2\cos\frac{\pi}{2} \\ &= 0 \end{aligned}$$

It is correct but not strictly necessary to give units for the acceleration, since zero acceleration means the same regardless of the units being used.

14. (a)
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 8\cos 2t \\ a_{|t=\frac{\pi}{6}} &= 8\cos\frac{\pi}{3} \\ &= 4\,\mathrm{m/s^2} \end{aligned}$$

(b)
$$\begin{aligned} x &= \int v\,dt \\ &= -2\cos 2t + c \\ 3 &= -2\cos 0 + c \\ c &= 5 \\ x &= 5 - 2\cos 2t \\ x_{|t=\frac{\pi}{2}} &= 5 - 2\cos\pi \\ &= 7\,\mathrm{m} \end{aligned}$$

15. This question is simplified if you first recognise that $4\sin t\cos t = 2\sin 2t$

(a)
$$\begin{aligned} v &= \int 2\sin 2t\,\mathrm{d}t \\ &= -\cos 2t + c \\ 3 &= -\cos 0 + c \\ c &= 4 \\ v &= 4 - \cos 2t \\ v_{|t=\frac{\pi}{3}} &= 4 - \cos\frac{2\pi}{3} \\ &= 4.5\,\mathrm{m/s} \end{aligned}$$

(b)
$$\begin{aligned} x &= 5 + \int_0^{\frac{\pi}{3}} (4 - \cos 2t)\,\mathrm{d}t \\ &= 5 + [4t - 0.5\sin 2t]_0^{\frac{\pi}{3}} \\ &= 5 + \left(\frac{4\pi}{3} - \frac{\sqrt{3}}{4}\right) - (0 - 0) \\ &= 5 + \frac{4\pi}{3} - \frac{\sqrt{3}}{4} \\ &\approx 8.756\,\mathrm{m} \end{aligned}$$

16.
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 18(3t+1)^2 \\ a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 108(3t+1) \end{aligned}$$

At $t = 2$,
$$\begin{aligned} x &= 2(3(2)+1)^3 \\ &= 686\,\mathrm{m} \\ v &= 18(3(2)+1)^2 \\ &= 882\,\mathrm{m/s} \\ a &= 108(3(2)+1) \\ &= 756\,\mathrm{m/s^2} \end{aligned}$$

17. (a)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{(2t+3) - 2(t+1)}{(2t+3)^2} \\ &= \frac{1}{(2t+3)^2} \\ a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= \frac{-4}{(2t+3)^3} \end{aligned}$$

(b) At $t = 1$,
$$\begin{aligned} x &= \frac{1+1}{2(1)+3} \\ &= \frac{2}{5} \\ &= 0.4\,\mathrm{m} \\ v &= \frac{1}{(2(1)+3)^2} \\ &= \frac{1}{25} \\ &= 0.04\,\mathrm{m/s} \\ a &= \frac{-4}{(2(1)+3)^3} \\ &= -\frac{4}{125} \\ &= -0.032\,\mathrm{m/s^2} \end{aligned}$$

18. (a)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= t^2 - 12t - 45 \\ &= (t-15)(t+3) \end{aligned}$$

when $v = 0$

$$\begin{aligned} t &= 15 \text{ (given that } t \geq 0) \\ x &= \frac{15^3}{3} - 6(15)^2 - 45(15) + 1000 \\ &= 100\,\mathrm{m} \end{aligned}$$

(b)
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 2t - 12 \end{aligned}$$
when $a = 0$
$$\begin{aligned} t &= 6 \\ x &= \frac{6^3}{3} - 6(6)^2 - 45(6) + 1000 \\ &= 586\,\text{m} \end{aligned}$$

19. "Hit the ground" means $h = 0$

$$\begin{aligned} 42 + 29t - 5t^2 &= 0 \\ t &= 7\,\text{s (ignoring the negative root)} \\ v &= \frac{\mathrm{d}h}{\mathrm{d}t} \\ &= 29 - 10t \\ v_{|t=7} &= 29 - 70 \\ &= -41\,\text{m/s} \end{aligned}$$

20. (a)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 16 - 2t \\ v_{|t=20} &= 16 - 40 \\ &= -24\,\text{m/s} \\ \therefore\ \text{speed} &= 24\,\text{m/s} \end{aligned}$$

(b)
$$\begin{aligned} v &= 0 \\ 16 - 2t &= 0 \\ t &= 8\,\text{s} \\ x &= 8(16 - 8) \\ &= 64\,\text{m} \end{aligned}$$

(c) Provided there is no change in direction, distance travelled is equal to the difference between the displacements:

$$\begin{aligned} d &= x_{|t=5} - x_{|t-1} \\ &= 5(16 - 5) - 1(16 - 1) \\ &= 55 - 15 \\ &= 40\,\text{m} \end{aligned}$$

(d) Between 5 and 10 seconds the particle reaches its maximum displacement and returns. We could find the distance travelled by integrating the absolute value of the velocity, but it's simpler to do it in two parts: from 5 to 8 seconds, and from 8 to 10 seconds.

$$\begin{aligned} d &= (64 - 55) + (64 - 10(16 - 10)) \\ &= 9 + (64 - 60) \\ &= 13\,\text{m} \end{aligned}$$

21.
$$\begin{aligned} v &= c - 9.8t \\ \text{initial velocity } v_{|t=0} &= c \\ \text{at max. height } v &= 0 \\ c - 9.8t &= 0 \\ c &= 9.8t \\ 250 &= (9.8t)t - 4.9t^2 \\ 250 &= 9.8t^2 - 4.9t^2 \\ 250 &= 4.9t^2 \\ t &= \sqrt{\frac{250}{4.9}} \\ c &= 9.8\sqrt{\frac{250}{4.9}} \\ &= 70\,\text{m/s} \end{aligned}$$

22.
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 3t^2 - 24t + 36 \\ d &= \int_1^8 |3t^2 - 24t + 36|\,\mathrm{d}t \\ &= 71\,\text{m} \end{aligned}$$

This could also be done without a calculator. First determine whether there is a change of direction in the interval of interest:

$$\begin{aligned} v &= 0 \\ 3t^2 - 24t + 36 &= 0 \\ t^2 - 8t + 12 &= 0 \\ (t - 2)(t - 6) &= 0 \end{aligned}$$

There are two changes of direction in the interval of interest so we need to find the distance in three sub-intervals: 1–2, 2–6 and 6–8:

$$\begin{aligned} d &= |x(2) - x(1)| \\ &\quad + |x(6) - x(2)| \\ &\quad + |x(8) - x(6)| \\ x(1) &= 1^3 - 12(1)^2 + 36(1) + 15 \\ &= 40 \\ x(2) &= 2^3 - 12(2)^2 + 36(2) + 15 \\ &= 47 \\ x(6) &= 6^3 - 12(6)^2 + 36(6) + 15 \\ &= 15 \\ x(8) &= 8^3 - 12(8)^2 + 36(8) + 15 \\ &= 47 \\ d &= 7 + 32 + 32 \\ &= 71\,\text{m} \end{aligned}$$

(You should know how to do this without a calculator, but your first impulse for a question like this is to use the calculator if it is available to you.)

23.
$$\begin{aligned}
v &= \int a\,\mathrm{d}t\\
&= 3t^2 - 24t + c\\
35 &= 3(0)^2 - 24(0) + c\\
c &= 35\\
v &= 3t^2 - 24t + 35\\
x &= \int v\,\mathrm{d}t\\
&= t^3 - 12t^2 + 35t + k\\
0 &= (0)^3 - 12(0)^2 + 35(0) + k\\
k &= 0\\
x &= t^3 - 12t^2 + 35t
\end{aligned}$$

solve for the first $x = 0, t > 0$

$$\begin{aligned}
t^3 - 12t^2 + 35t &= 0\\
t(t-5)(t-7) &= 0\\
t &= 5\\
v &= 3(5)^2 - 24(5) + 35\\
&= -10\,\mathrm{m/s}
\end{aligned}$$

24.
$$\begin{aligned}
v &= \int a\,\mathrm{d}t\\
&= 2t - t^2 + c\\
v_{|t=0} &= 24\\
\therefore\quad v &= 24 + 2t - t^2\\
x &= \int v\,\mathrm{d}t\\
&= 24t + t^2 - \frac{t^3}{3} + k\\
x_{|t=0} &= 0\\
\therefore\quad x &= 24t + t^2 - \frac{t^3}{3}
\end{aligned}$$

(a)
$$\begin{aligned}
v &= 0\\
24 + 2t - t^2 &= 0\\
(4+t)(6-t) &= 0\\
t &= 6\,\mathrm{s}\\
x &= 24(6) + (6)^2 - \frac{(6)^3}{3}\\
&= 144 + 36 - 72\\
&= 108\,\mathrm{m}
\end{aligned}$$

(b)
$$\begin{aligned}
x_{|t=3} &= 24(3) + (3)^2 - \frac{(3)^3}{3}\\
&= 72 + 9 - 9\\
&= 72\,\mathrm{m}\\
x_{|t=9} &= 24(9) + (9)^2 - \frac{(9)^3}{3}\\
&= 216 + 81 - 243\\
&= 54\,\mathrm{m}
\end{aligned}$$

(c)
$$\begin{aligned}
d &= (108 - 72) + (108 - 54)\\
&= 90\,\mathrm{m}
\end{aligned}$$

25. (a)
$$\begin{aligned}
v &= \int a\,\mathrm{d}t\\
&= 0.2t\\
x &= \int v\,\mathrm{d}t\\
&= 0.1t^2\\
x_{|t=180} &= 0.1(180)^2\\
&= 3240\,\mathrm{m}
\end{aligned}$$

(Both constants of integration are zero to account for the initial position and velocity both being zero.)

(b)
$$\begin{aligned}
v &= 0.2 \times 180\\
&= 36\,\mathrm{m/s}
\end{aligned}$$

26. "In the fourth second" means from $t = 3$ to $t = 4$:

$$\begin{aligned}
d &= \int_3^4 |7 + 2t|\,\mathrm{d}t\\
&= \int_3^4 (7 + 2t)\,\mathrm{d}t\\
&= \left[7t + t^2\right]_3^4\\
&= (28 + 16) - (21 + 9)\\
&= 14\,\mathrm{m}
\end{aligned}$$

(The absolute value can safely be dispensed with since v is positive for all positive t.)

27. "In the fourth second" means from $t = 3$ to $t = 4$. Here v changes sign at $t = 3.5$ so the absolute value must be retained. If doing the problem without a calculator, the integral should be divided into two parts, as follows:

$$\begin{aligned}
d &= \int_3^4 |7 - 2t|\,\mathrm{d}t\\
&= \int_3^{3.5} (7 - 2t)\,\mathrm{d}t + \int_{3.5}^4 (2t - 7)\,\mathrm{d}t\\
&= \left[7t - t^2\right]_3^{3.5} + \left[t^2 - 7t\right]_{3.5}^4\\
&= (24.5 - 12.25) - (21 - 9)\\
&\quad + (16 - 28) - (12.25 - 24.5)\\
&= 12.25 - 12 + (-12) - (-12.25)\\
&= 0.5\,\mathrm{m}
\end{aligned}$$

28. (a) The maximum value of $\sin 2t$ is 1, so the maximum velocity is $2 \times 1 = 2\,\mathrm{m/s}$.

(b)
$$\begin{aligned}
a &= \frac{\mathrm{d}v}{\mathrm{d}t}\\
&= 4\cos 2t
\end{aligned}$$

(c) The maximum value of $\cos 2t$ is 1, so the maximum acceleration is $4 \times 1 = 4\,\mathrm{m/s^2}$.

(d)
$$\begin{aligned}
x &= \int v\,\mathrm{d}t \qquad = -\cos 2t + c\\
0 &= -\cos 0 + c\\
c &= 1\\
x &= 1 - \cos 2t
\end{aligned}$$

(e) The maximum value of $-\cos 2t$ is 1, so the maximum displacement is $1 + 1 = 2\,\mathrm{m}$.

29. (a) The minimum value of $\sin^2 t$ is 0, so the minimum velocity is $3 \times 0 = 0\,\text{m/s}$. (I.e. the particle never moves backward.)

(b)
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 6\sin t\cos t \\ &= 3(2\sin t\cos t) \\ &= 3\sin 2t \end{aligned}$$

(c) The maximum value of $\sin 2t$ is 1 which first occurs when $2t = \frac{\pi}{2}$, i.e. $t = \frac{\pi}{4}$ s.

(d)
$$\begin{aligned} x &= \int v\,\mathrm{d}t \\ &= \int 3\sin^2 t\,\mathrm{d}t \\ &= -1.5\int (1 - 2\sin^2 t - 1)\,\mathrm{d}t \\ &= -1.5\int (\cos 2t - 1)\,\mathrm{d}t \\ &= -0.75\sin 2t + 1.5t + c \\ 0 &= -0.75\sin 0 + 1.5(0) + c \\ c &= 0 \\ x &= 1.5t - 0.75\sin 2t \end{aligned}$$

(e)
$$\begin{aligned} x &= 1.5\left(\frac{\pi}{6}\right) - 0.75\sin\left(\frac{\pi}{3}\right) \\ &= \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right)\,\text{m} \end{aligned}$$

30. (a)
$$\begin{aligned} \Delta x &= \int_0^4 3\sqrt{1+2t}\,\mathrm{d}t \\ &= \left[\frac{3(1+2t)^{1.5}}{1.5 \times 2}\right]_0^4 \\ &= \left[(1+2t)^{1.5}\right]_0^4 \\ &= 9^{1.5} - 1^{1.5} \\ &= 26\,\text{m} \end{aligned}$$

(b) Use the substitution

$$u = 1 + 2t \qquad t = \frac{u-1}{2}$$
$$du = 2dt \qquad dt = \frac{du}{2}$$

$$\begin{aligned} &\int_0^4 3t\sqrt{1+2t}\,\mathrm{d}t \\ &= \int_{1+2(0)}^{1+2(4)} \frac{3(u-1)}{2}\sqrt{u}\frac{du}{2} \\ &= \int_1^9 0.75(u-1)\sqrt{u}\,\mathrm{d}u \\ &= 0.75\int_1^9 u^{1.5} - u^{0.5}\,\mathrm{d}u \\ &= 0.75\left[\frac{u^{2.5}}{2.5} - \frac{u^{1.5}}{1.5}\right]_1^9 \\ &= \left[0.3u^{2.5} - 0.5u^{1.5}\right]_1^9 \\ &= \left(0.3(9)^{\frac{5}{2}} - 0.5(9)^{\frac{3}{2}}\right) \\ &\quad - \left(0.3(1)^{\frac{5}{2}} - 0.5(1)^{\frac{3}{2}}\right) \\ &= \left(0.3(3)^5 - 0.5(3)^3\right) - (0.3 - 0.5) \\ &= (0.3(243) - 0.5(27)) + 0.2 \\ &= 72.9 - 13.5 + 0.2 \\ &= 59.6\,\text{m} \end{aligned}$$

31.
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= \sqrt{3}\cos t + \sin t \end{aligned}$$

when $a = 0$

$$\begin{aligned} \sqrt{3}\cos t + \sin t &= 0 \\ \tan t &= -\sqrt{3} \\ t &= \frac{2\pi}{3} \\ \text{or } t &= \frac{5\pi}{3} \end{aligned}$$

$$\begin{aligned} x &= \int v\,\mathrm{d}t \\ &= -\sqrt{3}\sin t - \cos t + c \\ 0 &= -\sqrt{3}\sin\frac{\pi}{6} - \cos\frac{\pi}{6} + c \\ 0 &= -\sqrt{3}\left(\frac{1}{2}\right) - \frac{\sqrt{3}}{2} + c \\ 0 &= -\sqrt{3} + c \\ c &= \sqrt{3} \\ x &= \sqrt{3} - \sqrt{3}\sin t - \cos t \end{aligned}$$

when $t = \dfrac{2\pi}{3}$

$$\begin{aligned} x &= \sqrt{3} - \sqrt{3}\sin\frac{2\pi}{3} - \cos\frac{2\pi}{3} \\ &= \sqrt{3} - \frac{\sqrt{3}}{2} + \frac{1}{2} \\ &= \frac{\sqrt{3}+1}{2} \end{aligned}$$

$$\begin{aligned}\text{when } t &= \frac{5\pi}{3}\\ x &= \sqrt{3} - \sqrt{3}\sin\frac{5\pi}{3} - \cos\frac{5\pi}{3}\\ &= \sqrt{3} + \frac{\sqrt{3}}{2} - \frac{1}{2}\\ &= \frac{3\sqrt{3}-1}{2}\end{aligned}$$

32. (a)
$$\begin{aligned}v &= 3x + 2\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= 3\frac{\mathrm{d}x}{\mathrm{d}t}\\ a &= 3v\\ &= 3(3x+2)\\ &= (9x+6)\,\mathrm{m/s^2}\end{aligned}$$

(b)
$$\begin{aligned}v &= 3(4) + 2\\ &= 14\,\mathrm{m/s}\\ a &= 9(4) + 6\\ &= 42\,\mathrm{m/s^2}\end{aligned}$$

33. (a)
$$\begin{aligned}v &= 3x^2 - 2\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= 6x\frac{\mathrm{d}x}{\mathrm{d}t}\\ a &= 6xv\\ &= 6x(3x^2-2)\\ &= (18x^3 - 12x)\,\mathrm{m/s^2}\end{aligned}$$

(b)
$$\begin{aligned}v &= 3(1)^2 - 2\\ &= 1\,\mathrm{m/s}\\ a &= 18(1)^3 - 12(1)\\ &= 6\,\mathrm{m/s^2}\end{aligned}$$

Exercise 7C

1.
$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t} &= \frac{\mathrm{d}X}{\mathrm{d}t}\frac{\mathrm{d}p}{\mathrm{d}t}\\ &= (2\cos 2p)(2)\\ &= 4\cos 2p\\ &= 4\cos\frac{\pi}{3}\\ &= 2\end{aligned}$$

2.
$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\mathrm{d}A}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= 0.6\sin(3x)\cos(3x)\\ &= 0.3\sin 6x\\ &= 0.3\sin\frac{\pi}{6}\\ &= 0.15\end{aligned}$$

3.
$$\begin{aligned}y^2 &= 3x^3 + 1\\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9x^2\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{9x^2}{2y}\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= \frac{0.45x^2}{y}\end{aligned}$$

When $y = 5$
$$\begin{aligned}(5^2) &= 3x^3 + 1\\ 3x^3 &= 24\\ x^3 &= 8\\ x &= 2\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{0.45(2)^2}{5}\\ &= 0.36\end{aligned}$$

4.
$$\begin{aligned}x^2 + y^2 &= 400\\ 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y}\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= -6\frac{x}{y}\end{aligned}$$

When $y = 12$
$$\begin{aligned}x^2 + (12)^2 &= 400, x \geq 0\\ x &= \sqrt{400 - 144}\\ &= 16\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -6\frac{16}{12}\\ &= -8\end{aligned}$$

5. $$\begin{aligned} A &= \frac{1}{2}(10)(10)\sin x \\ &= 50\sin x \\ \frac{\mathrm{d}A}{\mathrm{d}x} &= 50\cos x \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\mathrm{d}A}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 0.5\cos x \\ &= 0.5\cos\frac{\pi}{3} \\ &= 0.25\,\mathrm{cm}^2/\mathrm{s} \end{aligned}$$

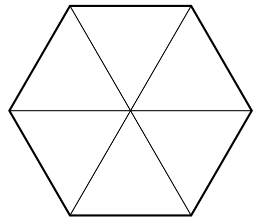
6. $$\begin{aligned} A &= \frac{1}{2}x^2\sin 45^\circ \\ &= \frac{\sqrt{2}\,x^2}{4} \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\sqrt{2}\,x}{2}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{0.1\sqrt{2}\,x}{2} \\ &= 0.05\sqrt{2}\,x \\ &= 0.05\sqrt{2}\,(10) \\ &= 0.5\sqrt{2} \\ &\approx 0.707\,\mathrm{cm}^2/\mathrm{s} \end{aligned}$$

7. $$\begin{aligned} x^2 + y^2 &= 10^2 \\ 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -0.1\frac{x}{y} \end{aligned}$$

   When $t = 20$
   $$\begin{aligned} x &= 4 + 0.1t \\ &= 6\,\mathrm{cm} \\ y &= \sqrt{100 - 6^2} \\ &= 8\,\mathrm{cm} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -0.1\frac{6}{8} \\ &= -0.075\,\mathrm{cm/s} \end{aligned}$$

8. Let x be the side length of the triangle.
   $$\begin{aligned} A &= \frac{1}{2}x^2\sin 60^\circ \\ &= \frac{\sqrt{3}\,x^2}{4} \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\sqrt{3}\,x}{2}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{\sqrt{3}\,(20)}{2}(0.2) \\ &= 2\sqrt{3}\,\mathrm{cm}^2/\mathrm{s} \end{aligned}$$

9. A regular hexagon can be divided into six equilateral triangles as shown.



   $$\begin{aligned} A &= 6 \times \frac{\sqrt{3}\,x^2}{4} \\ &= \frac{3\sqrt{3}\,x^2}{2} \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= 3\sqrt{3}\,x\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 3\sqrt{3}\,(20)(1) \\ &= 60\sqrt{3}\,\mathrm{cm}^2/\mathrm{minute} \\ &= \sqrt{3}\,\mathrm{cm}^2/\mathrm{s} \end{aligned}$$

10. From the cross-section being an equilateral triangle we have the relationship between height and radius:
   $$\begin{aligned} \tan 60^\circ &= \frac{h}{r} \\ h &= \sqrt{3}\,r \end{aligned}$$
   The volume is
   $$\begin{aligned} V &= \frac{\pi r^2 h}{3} \\ &= \frac{\pi\sqrt{3}\,r^3}{3} \end{aligned}$$
   Differentiating with respect to t
   $$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi\sqrt{3}\,r^2\frac{\mathrm{d}r}{\mathrm{d}t}$$
   and substitute for r and $\frac{\mathrm{d}r}{\mathrm{d}t}$
   $$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \pi\sqrt{3}\,(20)^2(0.5) \\ &= 200\pi\sqrt{3} \\ &\approx 1090\,\mathrm{cm}^3/\mathrm{s} \end{aligned}$$

11. Let x be the distance between the base of the ladder and the wall. Let y be the height of the top of the ladder above the ground.
   $$\begin{aligned} x^2 + y^2 &= 5.2^2 \\ 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -0.1\frac{x}{y} \\ x &= \sqrt{5.2^2 - 4.8^2} \\ &= 2.0 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -0.1\frac{2.0}{4.8} \\ &= -0.0417\,\mathrm{m/s} \end{aligned}$$

The top is moving down at approximately 4.2 cm/s.

12. (a) Let l be the length of the shadow and let d be the person's distance from the lamp-post. Using the similar triangles as a starting point,

$$\begin{aligned} \frac{l+d}{l} &= \frac{4.5}{1.5} \\ &= 3 \\ l+d &= 3l \\ d &= 2l \\ l &= 0.5d \\ \frac{\mathrm{d}l}{\mathrm{d}t} &= 0.5\frac{\mathrm{d}d}{\mathrm{d}t} \\ &= 1\,\mathrm{m/s} \end{aligned}$$

The shadow grows by 1 m/s.

(b) The tip of the shadow moves with the combined speed of the person and the increasing length of the shadow, i.e. $1+2=3$ m/s.

13.
$$\begin{aligned} r^2+(2-h)^2 &= 2^2 \\ 2r\frac{\mathrm{d}r}{\mathrm{d}t} - 2(2-h)\frac{\mathrm{d}h}{\mathrm{d}t} &= 0 \\ 2r\frac{\mathrm{d}r}{\mathrm{d}t} &= 2(2-h)\frac{\mathrm{d}h}{\mathrm{d}t} \\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{2-h}{r}\frac{\mathrm{d}h}{\mathrm{d}t} \\ \frac{\mathrm{d}h}{\mathrm{d}t} &= -0.005\,\mathrm{m/s} \\ h &= 1 \\ r &= \sqrt{2^2-(2-1)^2} \\ &= \sqrt{3} \\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{2-1}{\sqrt{3}}(-0.005) \\ &= -\frac{1}{200\sqrt{3}} \\ &= -\frac{\sqrt{3}}{6}00\,\mathrm{m/s} \\ &= -\frac{\sqrt{3}}{6}\,\mathrm{cm/s} \end{aligned}$$

The radius is decreasing at a rate of $\frac{\sqrt{3}}{6} \approx 0.29$ cm/s

14. Let BC$=x$ and AB$=y$.

$$\begin{aligned} y^2 &= x^2+20^2 \\ 2y\frac{\mathrm{d}y}{\mathrm{d}t} &= 2x\frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{48}{\sqrt{48^2+20^2}}(15) \\ &= \frac{48\times 15}{52} \\ &= \frac{180}{13} \\ &\approx 13.85\,\mathrm{ms}^{-1} \end{aligned}$$

15. Let d be the distance from A to the balloon and let h be the height of the balloon.

$$\begin{aligned} d^2 &= h^2+60^2 \\ 2d\frac{\mathrm{d}d}{\mathrm{d}t} &= 2h\frac{\mathrm{d}h}{\mathrm{d}t} \\ \frac{\mathrm{d}d}{\mathrm{d}t} &= \frac{h}{d}\frac{\mathrm{d}h}{\mathrm{d}t} \\ &= \frac{80}{\sqrt{80^2+60^2}}(5) \\ &= 4\,\mathrm{m/s} \end{aligned}$$

16.
$$\begin{aligned} x &= 8\tan\theta \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{8}{\cos^2\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ \cos^2\theta &= \left(\frac{8}{\sqrt{8^2+5^2}}\right)^2 \\ &= \frac{64}{89} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= (8)\left(\frac{89}{64}\right)(4\pi) \\ &= 44.5\pi \\ &\approx 139.8\,\mathrm{ms}^{-1} \end{aligned}$$

Note that it is not necessary to determine θ in order to find $\cos\theta$. Use the definition of cosine as $\frac{\text{adjacent}}{\text{hypotenuse}}$ to determine $\cos\theta$ directly.

17. Rotation of 5 revolutions per minute is 10π radians per minute or $\frac{\pi}{6}$ radians per second.

Let y be the distance along the coastline from the nearest point at time t. Let x be the straight line distance to the lighthouse at time t. We want $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $x=4$.

$$\begin{aligned} y^2+3^2 &= x^2 \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{y} \end{aligned}$$

$$\begin{aligned} \text{When } x=4, \quad y &= \sqrt{4^2-3^2} \\ &= \sqrt{7} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4}{\sqrt{7}} \end{aligned}$$

$$\begin{aligned}
\cos\theta &= \frac{3}{x}\\
x &= \frac{3}{\cos\theta}\\
\frac{dx}{d\theta} &= 3\frac{\sin\theta}{\cos^2\theta}\\
\text{When } x = 4, \quad \cos\theta &= \frac{3}{4}\\
\sin\theta &= \frac{\sqrt{7}}{4}\\
\frac{dx}{d\theta} &= 3\times\frac{\frac{\sqrt{7}}{4}}{\frac{9}{16}}\\
&= 3\times\frac{4\sqrt{7}}{9}\\
&= \frac{4\sqrt{7}}{3}\\
\frac{dy}{d\theta} &= \frac{dy}{dx}\frac{dx}{d\theta}\\
&= \left(\frac{4}{\sqrt{7}}\right)\left(\frac{4\sqrt{7}}{3}\right)\\
&= \frac{16}{3}\\
\frac{dy}{dt} &= \frac{dy}{d\theta}\frac{d\theta}{dt}\\
&= \left(\frac{16}{3}\right)\left(\frac{\pi}{6}\right)\\
&= \frac{8\pi}{9}\\
&\approx 2.79\,\text{km/s}
\end{aligned}$$

18.
$$\begin{aligned}
\tan\theta &= \frac{h}{600}\\
\frac{1}{\cos^2\theta}\frac{d\theta}{dt} &= \frac{1}{600}\frac{dh}{dt}\\
\frac{d\theta}{dt} &= \frac{\cos^2\theta}{600}\frac{dh}{dt}\\
\cos\theta &= \frac{600}{\sqrt{600^2+800^2}}\\
&= 0.6\\
\frac{d\theta}{dt} &= \frac{0.6^2}{600}\times 10\frac{dh}{dt}\\
&= 0.006 \text{ rads/sec}
\end{aligned}$$

19. Let x be the horizontal distance AB at time t.

$$\begin{aligned}
\tan\theta &= \frac{800}{x}\\
\frac{1}{\cos^2\theta}\frac{d\theta}{dt} &= -\frac{800}{x^2}\frac{dx}{dt}\\
\frac{d\theta}{dt} &= -\frac{800\cos^2\theta}{x^2}\frac{dx}{dt}\\
\cos^2\theta &= \left(\frac{1000}{\sqrt{1000^2+800^2}}\right)\\
&= \frac{25}{41}\\
\frac{d\theta}{dt} &= -\frac{800\times\frac{25}{41}}{1000^2}(-200)\\
&= \frac{4\,000\,000}{41\,000\,000}\\
&= \frac{4}{41} \text{ rads/s}
\end{aligned}$$

20.
$$\begin{aligned}
\tan\theta &= \frac{h}{200}\\
\frac{1}{\cos^2\theta}\frac{d\theta}{dt} &= \frac{1}{200}\frac{dh}{dt}\\
\frac{dh}{dt} &= \frac{200}{\cos^2\theta}\left(\frac{1}{20}\right)\\
&= \frac{10}{\cos^2\theta}\\
\frac{d^2h}{dt^2} &= -\frac{20(-\sin\theta)}{\cos^3\theta}\frac{d\theta}{dt}\\
&= \frac{20\tan\theta}{\cos^2\theta}\left(\frac{1}{20}\right)\\
&= \frac{\tan\theta}{\cos^2\theta}
\end{aligned}$$

(a) When $\theta = \frac{\pi}{6}$,

$$\begin{aligned}
v &= \frac{dh}{dt}\\
&= \frac{10}{\cos^2\theta}\\
&= \frac{10}{\left(\frac{\sqrt{3}}{2}\right)^2}\\
&= \frac{10}{\frac{3}{4}}\\
&= \frac{40}{3}\text{ ms}^{-1}\\
a &= \frac{d^2h}{dt^2}\\
&= \frac{\tan\theta}{\cos^2\theta}\\
&= \frac{\frac{\sqrt{3}}{3}}{\frac{3}{4}}\\
&= \frac{4\sqrt{3}}{9}\text{ ms}^{-2}
\end{aligned}$$

(b) When $\theta = \frac{\pi}{3}$,

$$\begin{aligned} v &= \frac{\mathrm{d}h}{\mathrm{d}t} \\ &= \frac{10}{\cos^2\theta} \\ &= \frac{10}{\left(\frac{1}{2}\right)^2} \\ &= \frac{10}{\frac{1}{4}} \\ &= 40\,\mathrm{ms}^{-1} \\ a &= \frac{\mathrm{d}^2h}{\mathrm{d}t^2} \\ &= \frac{\tan\theta}{\cos^2\theta} \\ &= \frac{\sqrt{3}}{\frac{1}{4}} \\ &= 4\sqrt{3}\,\mathrm{ms}^{-2} \end{aligned}$$

21. Let s be the length of the shadow CE and d be the distance AC. By similar triangles

$$\begin{aligned} \frac{s+d}{s} &= \frac{9}{1.8} \\ s+d &= 5s \\ d &= 4s \\ s &= 0.25d \\ \frac{\mathrm{d}s}{\mathrm{d}d} &= 0.25 \end{aligned}$$

$$\begin{aligned} d^2 &= 12^2 + y^2 \\ 2d\frac{\mathrm{d}d}{\mathrm{d}y} &= 2y \\ \frac{\mathrm{d}d}{\mathrm{d}y} &= \frac{y}{d} \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{\mathrm{d}s}{\mathrm{d}d}\frac{\mathrm{d}d}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (0.25)\left(\frac{y}{d}\right)(2) \\ &= \frac{y}{2d} \end{aligned}$$

When AC = 20, $y = 20$

$$\begin{aligned} d &= \sqrt{12^2+20^2} \\ &= 4\sqrt{34} \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{20}{8\sqrt{34}} \\ &= \frac{5}{2\sqrt{34}} \\ &\approx 0.43\,\mathrm{m/s} \end{aligned}$$

The shadow is growing 0.43 metres per second.

Exercise 7D

1–3 No working required.

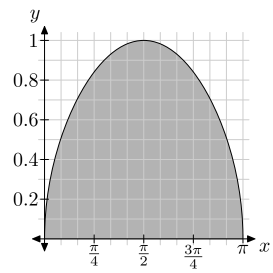
4.

$$\begin{aligned} u &= 2x \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= 2 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^{2x} 5^t dt\right) &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\int_1^{u} 5^t dt\right)\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (5^u)(2) \\ &= 2(5)^{2x} \end{aligned}$$

5.

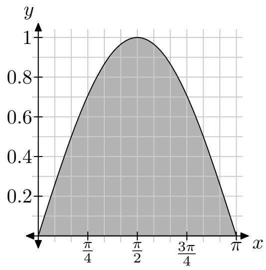
$$\begin{aligned} u &= 5x \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= 5 \\ \frac{\mathrm{d}}{\mathrm{d}x}&\left(\int_1^{5x} (3+4t+\sin t)dt\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\int_1^{u} (3+4t+\sin t)dt\right)\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (3+4u+\sin u)(5) \\ &= 5\left(3+20x+\sin(5x)\right) \\ &= 15+100x+5\sin(5x) \end{aligned}$$

6.



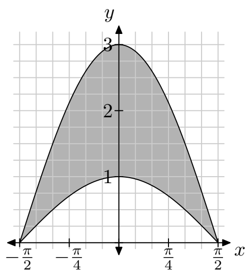
$$\begin{aligned} V &= \int_0^\pi \pi y^2 dx \\ &= \int_0^\pi \pi(\sqrt{\sin x})^2 dx \\ &= \int_0^\pi \pi\sin x\,\mathrm{d}x \\ &= [\pi(-\cos x)]_0^\pi \\ &= (-\pi\cos\pi) - (-\pi\cos 0) \\ &= \pi + \pi \\ &= 2\pi \text{ units}^3 \end{aligned}$$

7.



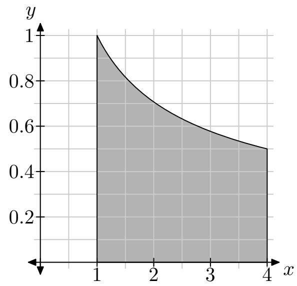
$$
\begin{aligned}
V &= \int_0^{\pi} \pi(\sin x)^2 dx \\
&= -\frac{\pi}{2}\int_0^{\pi} -2\sin^2 x \, \mathrm{d}x \\
&= -\frac{\pi}{2}\int_0^{\pi} (1-2\sin^2 x - 1)\, \mathrm{d}x \\
&= -\frac{\pi}{2}\int_0^{\pi} (\cos(2x)-1)\, \mathrm{d}x \\
&= -\frac{\pi}{2}\left[\frac{\sin 2x}{2} - x\right]_0^{\pi} \\
&= -\frac{\pi}{2}\left(\left(\frac{\sin 2\pi}{2} - \pi\right) - \left(\frac{\sin 0}{2} - 0\right)\right) \\
&= -\frac{\pi}{2}(-\pi - 0) \\
&= \frac{\pi^2}{2} \text{ units}^3
\end{aligned}
$$

8.



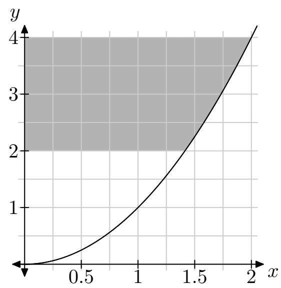
$$
\begin{aligned}
V &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi y_1^2 dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi y_2^2 dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi(y_1^2 - y_2^2) dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\left((3\cos x)^2 - (\cos x)^2\right) dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\pi\cos^2 x \, \mathrm{d}x \\
&= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos^2 x - 1 + 1) dx \\
&= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2x + 1) dx \\
&= 4\pi\left[\frac{\sin 2x}{2} + x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 4\pi\left(\left(\sin\frac{\pi}{2} + \frac{\pi}{2}\right) - \left(\sin\left(-\frac{\pi}{2}\right) - \frac{\pi}{2}\right)\right) \\
&= 4\pi\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \\
&= 4\pi^2 \text{ units}^3
\end{aligned}
$$

9.

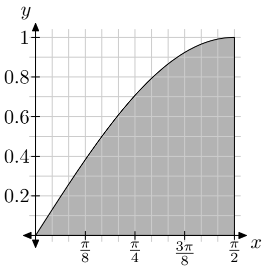


$$
\begin{aligned}
V &= \int_1^4 \pi\left(\frac{1}{\sqrt{x}}\right)^2 dx \\
&= \int_1^4 \frac{\pi}{x} dx \\
&= \pi\left[\ln x\right]_1^4 \\
&= \pi(\ln 4 - \ln 1) \\
&= \pi \ln 4 \text{ units}^3
\end{aligned}
$$

10.

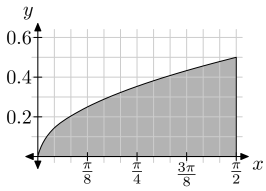


$$
\begin{aligned}
V &= \int_2^4 \pi x^2 dy \\
&= \int_2^4 \pi y \, dx \\
&= \left[\frac{\pi y^2}{2}\right]_2^4 \\
&= \frac{\pi}{2}(4^2 - 2^2) \\
&= 6\pi \text{ units}^3
\end{aligned}
$$



11. First possibility:

$$\begin{aligned}V &= \int_0^{\frac{\pi}{2}} \pi \left(\frac{\sin x}{2}\right)^2 dy \\ &= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \sin^2 x \,\mathrm{d}y \\ &= -\frac{\pi}{8}\int_0^{\frac{\pi}{2}} (1 - 2\sin^2 x - 1)\,\mathrm{d}y \\ &= -\frac{\pi}{8}\int_0^{\frac{\pi}{2}} (\cos 2x - 1)\,\mathrm{d}y \\ &= -\frac{\pi}{8}\left[\frac{\sin 2x}{2} - x\right]_0^{\frac{\pi}{2}} \\ &= -\frac{\pi}{8}\left(-\frac{\pi}{2} - 0\right) \\ &= \frac{\pi^2}{16}\,\mathrm{m}^3\end{aligned}$$



Second possibility:

$$\begin{aligned}V &= \int_0^{\frac{\pi}{2}} \pi \left(\sqrt{\frac{x}{2\pi}}\right)^2 dy \\ &= \int_0^{\frac{\pi}{2}} \pi \frac{x}{2\pi} dy \\ &= \int_0^{\frac{\pi}{2}} \frac{x}{2} dy \\ &= \left[\frac{x^2}{4}\right]_0^{\frac{\pi}{2}} \\ &= (\frac{\pi^2}{16} - 0) \\ &= \frac{\pi^2}{16}\,\mathrm{m}^3\end{aligned}$$

Chapter 7 Extension Activity

1. The y-coordinate of the centre of gravity is 0 from the symmetry of the shape.

   Sum of moments is

$$\begin{aligned}\int_0^r mg2yx\,\mathrm{d}x &= -mg\int_0^r -2x\sqrt{r^2 - x^2}\,\mathrm{d}x \\ &= -mg\left[\frac{2}{3}(r^2 - x^2)^{\frac{3}{2}}\right]_0^r \\ &= -\frac{2}{3}mg\left((0)^{\frac{3}{2}} - (r^2)^{\frac{3}{2}}\right) \\ &= \frac{2}{3}mgr^3\end{aligned}$$

   The moment of the sum is

$$mg\frac{\pi r^2}{2}(\bar{x})$$

   Thus

$$\begin{aligned}mg\frac{\pi r^2}{2}(\bar{x}) &= \frac{2}{3}mgr^3 \\ \frac{\pi}{2}(\bar{x}) &= \frac{2}{3}r \\ \bar{x} &= \frac{4r}{3\pi}\end{aligned}$$

   $\therefore$ the centre of gravity is at the point $\left(\frac{4r}{3\pi}, 0\right)$.

2. The y-coordinate of the centre of gravity is 0 from the symmetry of the shape.

   The line segment AB is on the line passing through the origin and (a, b), i.e. $y = \frac{b}{a}x$.

   Sum of moments is

$$\begin{aligned}\int_0^a mg2yx\,\mathrm{d}x &= 2mg\int_0^a \left(\frac{b}{a}x\right)x\,\mathrm{d}x \\ &= \frac{2mgb}{a}\int_0^a x^2\,\mathrm{d}x \\ &= \frac{2mgb}{a}\left[\frac{x^3}{3}\right]_0^a \\ &= \frac{2mgb}{3a}\left[x^3\right]_0^a \\ &= \frac{2mgb}{3a}\left(a^3 - 0^3\right) \\ &= \frac{2mga^2b}{3}\end{aligned}$$

   The moment of the sum is

$$mgab\bar{x}$$

Thus

$$\begin{aligned} mgab\bar{x} &= \frac{2mga^2b}{3} \\ \bar{x} &= \frac{2a}{3} \end{aligned}$$

$\therefore$ the centre of gravity is at the point $\left(\frac{2a}{3}, 0\right)$.

3. Area of one strip:

$$\begin{aligned} A_x &\approx ((0.5x+2) - (-0.5x-2))\,\delta x \\ &= (x+4)\delta x \end{aligned}$$

Moment of one strip:

$$I_x \approx mgx(x+4)\delta x$$

Sum of moments:

$$\begin{aligned} I &= \int_2^6 mgx(x+4)\,\mathrm{d}x \\ &= mg\left[\frac{x^3}{3} + 2x^2\right]_2^6 \\ &= mg\left(\left(\frac{6^3}{3} + 2(6)^2\right)\right. \\ &\qquad \left.-\left(\frac{2^3}{3} + 2(2)^2\right)\right) \\ &= mg\left((72+72) - \left(\frac{8}{3}+8\right)\right) \\ &= \frac{400}{3}mg \end{aligned}$$

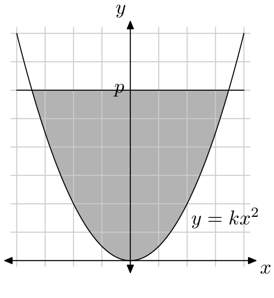
Total area:

$$\begin{aligned} A &= \frac{1}{2}\left((2+4)+(6+4)\right)(6-2) \\ &= 2\,(6+10) \\ &= 32 \end{aligned}$$

Moment of sum:

$$\begin{aligned} I &= 32mg\bar{x} \\ \therefore \quad 32mg\bar{x} &= \frac{400}{3}mg \\ \bar{x} &= \frac{25}{6} \end{aligned}$$

The symmetry of the figure gives us $\bar{y} = 0$ so the centre of gravity is at $\left(\frac{25}{6}, 0\right)$.

4. 

From the symmetry of the figure, $\bar{x} = 0$.

Taking horizontal strips of height δy and the moment of inertia about the $x-$axis,

$$\begin{aligned} y &= kx^2 \\ x &= \pm\sqrt{\frac{y}{k}} \end{aligned}$$

Area of one strip:

$$A_y \approx 2\sqrt{\frac{y}{k}}\delta y$$

Moment of one strip:

$$\begin{aligned} I_y &\approx mgy\left(2\sqrt{\frac{y}{k}}\right)\delta y \\ &= \frac{2mg}{\sqrt{k}}y^{\frac{3}{2}}\delta y \end{aligned}$$

Sum of moments:

$$\begin{aligned} I &= \int_0^p \frac{2mg}{\sqrt{k}}y^{\frac{3}{2}}\,\mathrm{d}y \\ &= \frac{2mg}{\sqrt{k}}\left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^p \\ &= \frac{4mg}{5\sqrt{k}}\left[y^{\frac{5}{2}}\right]_0^p \\ &= \frac{4mgp^{\frac{5}{2}}}{5\sqrt{k}} \end{aligned}$$

Total area:

$$\begin{aligned} A &= \int_0^p 2\sqrt{\frac{y}{k}}\,\mathrm{d}y \\ &= \frac{2}{\sqrt{k}}\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^p \\ &= \frac{4p^{\frac{3}{2}}}{3\sqrt{k}} \end{aligned}$$

Moment of sum:

$$\begin{aligned} I &= \frac{4p^{\frac{3}{2}}}{3\sqrt{k}}mg\bar{y} \\ &= \frac{4mgp^{\frac{3}{2}}}{3\sqrt{k}}\bar{y} \\ \therefore \quad \frac{4mgp^{\frac{3}{2}}}{3\sqrt{k}}\bar{y} &= \frac{4mgp^{\frac{5}{2}}}{5\sqrt{k}} \\ \bar{y} &= \frac{3p}{5} \end{aligned}$$

Hence the centre of gravity is at $\left(0, \frac{3p}{5}\right)$.

5. From the symmetry of the figure, $\bar{y} = 0$.

The radius of one disc is

$$y = \sqrt{r^2 - x^2}$$

Volume of one disc:

$$\begin{aligned} V_x &\approx \pi y^2 \delta x \\ &= \pi(r^2 - x^2)\delta x \end{aligned}$$

Moment of one strip (where m is mass per unit volume):

$$\begin{aligned} I_x &\approx mgx\pi(r^2 - x^2)\delta x \\ &= \pi mg(r^2 x - x^3)\delta x \end{aligned}$$

Sum of moments:

$$\begin{aligned} I &= \int_0^r \pi mg(r^2 x - x^3)\,\mathrm{d}x \\ &= \pi mg \left[\frac{r^2x^2}{2} - \frac{x^4}{4}\right]_0^r \\ &= \frac{\pi mg}{4}\left[2r^2x^2 - x^4\right]_0^r \\ &= \frac{\pi mgr^4}{4} \end{aligned}$$

Total volume:

$$V = \frac{2\pi r^3}{3}$$

Moment of sum:

$$\begin{aligned} I &= \frac{2\pi r^3}{3} mg\bar{x} \\ &= \frac{2\pi mgr^3}{3}\bar{x} \\ \therefore \quad \frac{2\pi mgr^3}{3}\bar{x} &= \frac{\pi mgr^4}{4} \\ \bar{x} &= \frac{3r}{8} \end{aligned}$$

Hence the centre of gravity is at $\left(\frac{3r}{8}, 0\right)$.

6. From the symmetry of the figure, $\bar{y} = 0$.

   Slice the figure into vertical discs of thickness δx. The radius of one disc is

$$y = \frac{xr}{h}$$

Volume of one disc:

$$\begin{aligned} V_x &\approx \pi y^2 \delta x \\ &= \pi\left(\frac{xr}{h}\right)^2 \delta x \\ &= \frac{\pi r^2 x^2}{h^2}\delta x \end{aligned}$$

Moment of one strip (where m is mass per unit volume):

$$\begin{aligned} I_x &\approx mgx\left(\frac{\pi r^2 x^2}{h^2}\right)\delta x \\ &= \left(\frac{\pi mgr^2x^3}{h^2}\right)\delta x \end{aligned}$$

Sum of moments:

$$\begin{aligned} I &= \int_0^h \frac{\pi mgr^2x^3}{h^2}\,\mathrm{d}x \\ &= \frac{\pi mgr^2}{h^2}\left[\frac{x^4}{4}\right]_0^h \\ &= \frac{\pi mgr^2h^2}{4} \end{aligned}$$

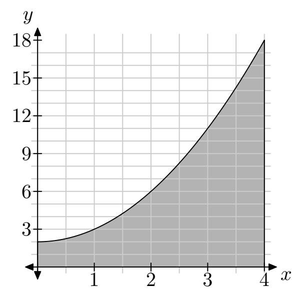
Total volume:

$$V = \frac{\pi r^2 h}{3}$$

Moment of sum:

$$\begin{aligned} I &= \frac{\pi r^2 h}{3} mg\bar{x} \\ &= \frac{\pi mgr^2 h}{3}\bar{x} \\ \therefore \quad \frac{\pi mgr^2h}{3}\bar{y} &= \frac{\pi mgr^2h^2}{4} \\ \bar{y} &= \frac{3h}{4} \end{aligned}$$

Hence the centre of gravity is at $\left(\frac{3h}{4}, 0\right)$.

7. 

Divide the shape into vertical strips of width δx. Area of one strip:

$$A_x \approx \left(x^2 + 2\right)\delta x$$

Moment of one strip about the y-axis:

$$I_x \approx mgx(x^2 + 2)\delta x$$

Sum of moments:

$$\begin{aligned} I &= \int_0^4 mgx(x^2+2)\,\mathrm{d}x \\ &= mg\int_0^4 (x^3 + 2x)\,\mathrm{d}x \\ &= mg\left[\frac{x^4}{4} + x^2\right]_0^4 \\ &= mg(64 + 16) \\ &= 80mg \end{aligned}$$

Total area:

$$\begin{aligned} A &= \int_0^4 (x^2+2)\,\mathrm{d}x \\ &= \left[\frac{x^3}{3}+2x\right]_0^4 \\ &= \frac{64}{3}+8 \\ &= \frac{88}{3} \end{aligned}$$

Moment of sum:

$$\begin{aligned} I &= \frac{88}{3}mg\bar{x} \\ \therefore \quad \frac{88}{3}mg\bar{x} &= 80mg \\ \bar{x} &= \frac{240}{88} \\ &= \frac{30}{11} \end{aligned}$$

Now consider the moment of each strip about the x-axis. The centre of gravity of each strip is $0.5y$ from the x-axis (as the hint states) so the moment of each strip is

$$\begin{aligned} I_x &\approx mg\frac{y}{2}(x^2+2)\delta x \\ &= \frac{mg}{2}(x^2+2)(x^2+2)\delta x \\ &= \frac{mg}{2}(x^4+4x^2+4)\delta x \end{aligned}$$

Sum of moments:

$$\begin{aligned} I &= \int_0^4 \frac{mg}{2}(x^4+4x^2+4)\,\mathrm{d}x \\ &= \frac{mg}{2}\left[\frac{x^5}{5}+\frac{4x^3}{3}+4x\right]_0^4 \\ &= \frac{mg}{2}\left(\frac{1024}{5}+\frac{256}{3}+16\right) \\ &= \frac{2296mg}{15} \end{aligned}$$

Moment of sum:

$$\begin{aligned} I &= mgA\bar{y} \\ &= \frac{88mg}{3}\bar{y} \therefore \quad \frac{88mg}{3}\bar{y} = \frac{2296mg}{15} \\ \bar{y} &= \frac{287}{55} \end{aligned}$$

The centre of gravity is $(\bar{x},\bar{y}) = \left(\frac{30}{11}, \frac{287}{55}\right)$.

Miscellaneous Exercise 7

1-2 What working there is for this question is trivial enough (I hope) to not need further elucidation here.

3. (a)

$$\begin{aligned} 2|x|-3 &< 0 \\ 2|x| &< 3 \\ |x| &< \frac{3}{2} \\ x &< \frac{3}{2} \\ \text{and } -x &< -\frac{3}{2} \\ \therefore \quad -\frac{3}{2} < x &< \frac{3}{2} \end{aligned}$$

(b) First consider the case $2x-3 \geq 0$ so $|2x-3| = 2x-3$. It also follows that $x \geq \frac{3}{2}$ so $|x| = x$. The inequality then gives

$$\begin{aligned} 2|x|-3 &< |2x-3| \\ 2x-3 &< 2x-3 \end{aligned}$$

This is clearly a contradiction and so we must exclude $x \geq \frac{3}{2}$ from the solution.

Next consider the case $2x-3<0$ and $x \geq 0$. This gives $|2x-3| = -(2x-3)$ and $|x| = x$. Hence

$$\begin{aligned} 2|x|-3 &< |2x-3| \\ 2x-3 &< -(2x-3) \\ 2x-3 &< -2x+3 \\ 4x &< 6 \\ x &< \frac{3}{2} \end{aligned}$$

Finally consider the case $2x-3<0$ and $x<0$. This gives $|2x-3| = -(2x-3)$ and $|x| = -x$. Hence

$$\begin{aligned} 2|x|-3 &< |2x-3| \\ -2x-3 &< -(2x-3) \\ -2x-3 &< -2x+3 \\ -3 &< 3 \end{aligned}$$

This is true regardless of the value of x. Combining these results gives us the solution $x < \frac{3}{2}$.

4. (a) Consider the three possible cases: $x > 3$, $1 < x \leq 3$ and $x \leq 1$.
For $x > 3$, $x - 3 > 0$ so $|x-3| = x-3$ and $x - 1 > 0$ so $|x-1| = x-1$;

$$\begin{aligned} |x-3| + |x-1| &= 4 \\ x - 3 + x - 1 &= 4 \\ 2x - 4 &= 4 \\ x &= 4 \end{aligned}$$

For $1 < x \leq 3$, $x - 3 \leq 0$ so $|x-3| = -x+3$ and $x - 1 > 0$ so $|x-1| = x-1$;

$$\begin{aligned} |x-3| + |x-1| &= 4 \\ -x + 3 + x - 1 &= 4 \\ 2 &= 4 \end{aligned}$$

No solution.
For $x \leq 1$, $x - 3 < 0$ so $|x-3| = -x+3$ and $x - 1 \leq 0$ so $|x-1| = -x+1$;

$$\begin{aligned} |x-3| + |x-1| &= 4 \\ -x + 3 - x + 1 &= 4 \\ -2x + 4 &= 4 \\ x &= 0 \end{aligned}$$

The two solutions are $x = 0$ and $x = 4$.

(b) Again, consider the three possible cases: $x > 3$, $1 < x \leq 3$ and $x \leq 1$.
For $x > 3$, $x - 3 > 0$ so $|x-3| = x-3$ and $x - 1 > 0$ so $|x-1| = x-1$;

$$\begin{aligned} |x-3| + |x-1| &= 2 \\ x - 3 + x - 1 &= 2 \\ 2x - 4 &= 2 \\ x &= 3 \end{aligned}$$

Strictly speaking this is not in the part of the domain we are considering (but $x = 3$ *is* a solution as shown by the next part.)
For $1 < x \leq 3$, $x - 3 \leq 0$ so $|x-3| = -x+3$ and $x - 1 > 0$ so $|x-1| = x-1$;

$$\begin{aligned} |x-3| + |x-1| &= 2 \\ -x + 3 + x - 1 &= 2 \\ 2 &= 2 \end{aligned}$$

This is true for all $1 < x \leq 3$
For $x \leq 1$, $x - 3 < 0$ so $|x-3| = -x+3$ and $x - 1 \leq 0$ so $|x-1| = -x+1$;

$$\begin{aligned} |x-3| + |x-1| &= 2 \\ -x + 3 - x + 1 &= 2 \\ -2x + 4 &= 2 \\ -2x &= -2 \\ x &= 1 \end{aligned}$$

The two solution is $1 \leq x \leq 3$.

5.
$$\begin{aligned} \frac{\delta y}{\delta x} &\approx \frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \sin x + x\cos x \\ \delta y &\approx \delta x(\sin x + x\cos x) \\ &= 0.05(\sin 2.5 + 2.5\cos 2.5) \\ &= -0.07 \end{aligned}$$

6.
$$\begin{aligned} A &= ACC^{-1} \\ &= \begin{bmatrix} 2 & -4 \\ 11 & 3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.1 \\ -0.4 & 0.3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \\ B &= C^{-1}CB \\ &= \begin{bmatrix} 0.2 & 0.1 \\ -0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 6 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

7.
$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} &= \begin{bmatrix} 4 & 9 \\ 1 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}^{-1} &= \frac{1}{-3+2} \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 4 & 9 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

$a = 3, b = -1, c = 2, d = 1$

8.
$$\begin{aligned} AB &= \begin{bmatrix} 5 & 2 \\ x & y \end{bmatrix} \begin{bmatrix} 1 & 6 \\ -3 & z \end{bmatrix} \\ &= \begin{bmatrix} -1 & 30 + 2z \\ x - 3y & 6x + yz \end{bmatrix} \\ BA &= \begin{bmatrix} 1 & 6 \\ -3 & z \end{bmatrix} \begin{bmatrix} 5 & 2 \\ x & y \end{bmatrix} \\ &= \begin{bmatrix} 5 + 6x & 2 + 6y \\ -15 + xz & -6 + yz \end{bmatrix} \\ AB &= BA \\ \begin{bmatrix} -1 & 30 + 2z \\ x - 3y & 6x + yz \end{bmatrix} &= \begin{bmatrix} 5 + 6x & 2 + 6y \\ -15 + xz & -6 + yz \end{bmatrix} \\ 5 + 6x &= -1 \\ x &= -1 \\ \begin{bmatrix} -1 & 30 + 2z \\ -1 - 3y & -6 + yz \end{bmatrix} &= \begin{bmatrix} -1 & 2 + 6y \\ -15 - z & -6 + yz \end{bmatrix} \\ 30 + 2z &= 2 + 6y \\ 2z &= -28 + 6y \\ z &= 3y - 14 \\ \text{check:}\quad -1 - 3y &= -15 - z \\ 14 - 3y &= -z \\ z &= 3y - 14 \end{aligned}$$

9. No working required.

10.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x\ln x + x^2\frac{1}{x}\\ &= 2x\ln x + x\\ &= x(2\ln x + 1)\end{aligned}$$

11-12 No working required.

13. (a)
$$\begin{aligned}e^{\frac{\pi i}{2}} &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\\ &= 0 + i\\ &= (0,1)\end{aligned}$$

(b)
$$\begin{aligned}e^{(2-0.5\pi i)} &= (e^2)(e^{0.5\pi i})\\ &= e^2\left(\cos(-0.5\pi) + i\sin(-0.5\pi)\right)\\ &= e^2(0 - i)\\ &= (0, -e^2)\end{aligned}$$

(c)
$$\begin{aligned}e^2 + 4e^{\frac{\pi i}{3}} &= e^2 + 4\cos\frac{\pi}{3} + 4i\sin\frac{\pi}{3}\\ &= e^2 + 2 + 2\sqrt{3}i\\ &= (e^2 + 2, 2\sqrt{3})\end{aligned}$$

14.
$$\begin{aligned}\int\frac{dA}{A} &= \int -0.02\,\mathrm{d}t\\ \ln A &= -0.02t + c\\ A &= A_0e^{-0.02t}\\ \frac{A}{A_0} &= e^{-0.02t}\\ e^{-0.02t_h} &= 0.5\\ -0.02t_h &= \ln(0.5)\\ t_h &= \frac{\ln(0.5)}{-0.02}\\ &\approx 34.66 \text{ years}\end{aligned}$$

15. (a)
$$\begin{aligned}u &= 3x^2 - 5\\ du &= 6x\,\mathrm{d}x\\ \int x(3x^2-5)^7\,\mathrm{d}x &= \int u^7\,\frac{du}{6}\\ &= \frac{u^8}{8\times 6} + c\\ &= \frac{(3x^2-5)^8}{48} + c\end{aligned}$$

(b)
$$\begin{aligned}u &= x - 5\\ x &= u + 5\\ du &= dx\\ \int x(x-5)^7\,\mathrm{d}u &= \int (u+5)(u)^7\,\mathrm{d}u\\ &= \int (u^8 + 5u^7)\,\mathrm{d}u\\ &= \frac{u^9}{9} + \frac{5u^8}{8} + c\\ &= \frac{u^8}{72}(8u + 45) + c\\ &= \frac{(x-5)^8}{72}(8(x-5) + 45) + c\\ &= \frac{(x-5)^8}{72}(8x - 40 + 45) + c\\ &= \frac{(x-5)^8}{72}(8x - 5) + c\end{aligned}$$

(c)
$$\begin{aligned}u &= x^2 - 3\\ du &= 2\,\mathrm{d}x\\ \int\frac{8x}{\sqrt{x^2-3}}dx &= \int\frac{4}{\sqrt{u}}du\\ &= \frac{4\sqrt{u}}{0.5} + c\\ &= 8\sqrt{x^2-3} + c\end{aligned}$$

(d)
$$\begin{aligned}u &= 5x - 2\\ x &= \frac{u+2}{5}\\ du &= 5\,\mathrm{d}x\end{aligned}$$
$$\begin{aligned}&\int 10x\sqrt{5x-2}\,\mathrm{d}x\\ &= \int 2(u+2)\sqrt{u}\frac{du}{5}\\ &= \frac{2}{5}\int (u^{\frac{3}{2}} + 2u^{\frac{1}{2}})\,\mathrm{d}u\\ &= \frac{2}{5}\left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}(2u^{\frac{3}{2}})\right) + c\\ &= \frac{4}{75}u^{\frac{3}{2}}(3u + 10) + c\\ &= \frac{4}{75}(5x-2)^{\frac{3}{2}}(3(5x-2) + 10) + c\\ &= \frac{4}{75}(5x-2)^{\frac{3}{2}}(15x - 6 + 10) + c\\ &= \frac{4}{75}(5x-2)^{\frac{3}{2}}(15x + 4) + c\end{aligned}$$

(e)
$$\begin{aligned}u &= x^2 - 5\\ du &= 2x\,\mathrm{d}x\\ \int 8x\sin(x^2-5)\,\mathrm{d}x &= \int 4\sin u\,\mathrm{d}u\\ &= -4\cos u + c\\ &= -4\cos(x^2-5) + c\end{aligned}$$

(f)
$$\begin{aligned}u &= 1 + e^x\\ \mathrm{d}u &= e^x\,\mathrm{d}x\\ \int e^x(1+e^x)^4\,\mathrm{d}x &= \int u^4\,\mathrm{d}u\\ &= \frac{u^5}{5} + c\\ &= \frac{1+e^x}{5} + c\end{aligned}$$

(g)

$$\begin{aligned}
u &= x-3\\
x &= u+3\\
du &= dx\\
\int \frac{4x}{\sqrt{x-3}}dx &= \int \frac{4(u+3)}{\sqrt{u}}du\\
&= \int (4u^{0.5}+12u^{-0.5})du\\
&= \frac{4u^{1.5}}{1.5}+\frac{12u^{0.5}}{0.5}+c\\
&= \frac{8u^{1.5}}{3}+24u^{0.5}+c\\
&= \frac{8u^{1.5}+72u^{0.5}}{3}+c\\
&= \frac{8}{3}\sqrt{u}(u+9)+c\\
&= \frac{8}{3}\sqrt{x-3}(x-3+9)+c\\
&= \frac{8}{3}\sqrt{x-3}(x+6)+c
\end{aligned}$$

(h)

$$\begin{aligned}
u &= x+2\\
x &= u-2\\
2x &= 2u-4\\
2x+1 &= 2u-3\\
du &= dx\\
\int \frac{2x+1}{(x+2)^3}dx &= \int \frac{2u-3}{u^3}dx\\
&= \int (2u^{-2}-3u^{-3})dx\\
&= \frac{2u^{-1}}{-1}-\frac{3u^{-2}}{-2}+c\\
&= -2u^{-1}+1.5u^{-2}+c\\
&= \frac{1.5-2u}{u^2}+c\\
&= \frac{1.5-2(x+2)}{(x+2)^2}+c\\
&= \frac{1.5-2x-4}{(x+2)^2}+c\\
&= \frac{-2x-2.5}{(x+2)^2}+c\\
&= -\frac{4x-5}{2(x+2)^2}+c
\end{aligned}$$

(i)

$$\begin{aligned}
u &= 2^x\\
&= e^{\ln 2^x}\\
&= e^{x\ln 2}\\
du &= \ln(2)e^{x\ln 2}\,\mathrm{d}x\\
&= \ln(2)\,2^x\,\mathrm{d}x\\
\int 2^x\,\mathrm{d}x &= \int \frac{du}{\ln 2}\\
&= \frac{u}{\ln 2}+c\\
&= \frac{2^x}{\ln 2}+c
\end{aligned}$$

(j)

$$\begin{aligned}
u &= 5^{x+1}\\
du &= \ln(5)\,5^{x+1}\,\mathrm{d}x\\
\int 5^{x+1}\,\mathrm{d}x &= \int \frac{du}{\ln 5}\\
&= \frac{u}{\ln 5}+c\\
&= \frac{5^{x+1}}{\ln 5}+c
\end{aligned}$$

16.

$$\begin{aligned}
\frac{\mathrm{d}M}{\mathrm{d}t} &= -kM\\
\int \frac{dM}{M} &= \int k\,\mathrm{d}t\\
\ln M &= kt+c\\
M &= M_0\mathrm{e}^{kt}\\
0.5 &= \mathrm{e}^{1600k}\\
1600k &= -\ln 2\\
k &= \frac{-\ln 2}{1600}\\
&\approx -0.000433\\
M_0 &= 5000\,\mathrm{g}\\
M &= 5000\mathrm{e}^{-0.000433t}\ \mathrm{g}
\end{aligned}$$

After 100 years,

$$\begin{aligned}
M &= 5000\mathrm{e}^{-0.000433\times 100}\\
&= 5000\mathrm{e}^{-0.0433}\\
&\approx 4\,788\,\mathrm{g}
\end{aligned}$$

17. (a)

$$\begin{aligned}
\int_0^{\frac{2\pi}{3}} \sin x\,\mathrm{d}x &= [-\cos x]_0^{\frac{2\pi}{3}}\\
&= \frac{1}{2}+1\\
&= 1.5
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{0} \sin 2x\,\mathrm{d}x &= \left[\frac{-\cos 2x}{2}\right]_{-\frac{\pi}{2}}^{0}\\
&= -\frac{1}{2}-\frac{1}{2}\\
&= -1
\end{aligned}$$

(c)

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,\mathrm{d}x\\
&= -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-2\sin^2 x-1)\,\mathrm{d}x\\
&= -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2x-1)\,\mathrm{d}x\\
&= -\frac{1}{2}\left[\frac{\sin 2x}{2}-x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\
&= -\frac{1}{4}\left[\sin(2x)-2x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\
&= -\frac{1}{4}\left((\sin(\pi)-\pi)-(\sin(-\pi)+\pi)\right)\\
&= -\frac{1}{4}(-\pi-\pi)\\
&= \frac{\pi}{2}
\end{aligned}$$

18. (a)

$$\begin{aligned}\int_1^3 x^a\,\mathrm{d}x &= \left[\frac{x^{a+1}}{a+1}\right]_1^3\\ &= \frac{3^{a+1}}{a+1} - \frac{1^{a+1}}{a+1}\\ &= \frac{(3)3^a - 1}{a+1}\end{aligned}$$

(b)

$$\begin{aligned}\int_2^a 6x^2\,\mathrm{d}x &= \left[2x^3\right]_2^a\\ &= 2a^3 - 16\end{aligned}$$

(c)

$$\begin{aligned}\text{Let } u &= \sin 2x\\ \mathrm{d}u &= 2\cos 2x\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{\pi}{12}} \sin^a 2x\cos 2x\,\mathrm{d}x &= \int_0^{\sin\frac{\pi}{6}} u^a\,\frac{\mathrm{d}u}{2}\\ &= \int_0^{\frac{1}{2}} \frac{u^a}{2}\,\mathrm{d}u\\ &= \left[\frac{u^{a+1}}{2(a+1)}\right]_0^{\frac{1}{2}}\\ &= \left(\frac{\left(\frac{1}{2}\right)^{a+1}}{2(a+1)}\right)\\ &= \frac{1}{2(a+1)2^{a+1}}\\ &= \frac{1}{(a+1)2^{a+2}}\end{aligned}$$

19. (a)

$$\begin{aligned}d &= \int_0^{0.5} |v|\,\mathrm{d}t\\ &= \int_0^{0.5} |5\cos 2t|\,\mathrm{d}t\end{aligned}$$

$$5\cos 2t > 0\ \forall 0 \le t \le 0.5$$

$$\begin{aligned}\therefore\quad d &= \int_0^{0.5} 5\cos(2t)\,\mathrm{d}t\\ &= \left[\frac{5}{2}\sin 2t\right]_0^{0.5}\\ &= 2.5\sin 1\\ &\approx 2.10\text{m}\end{aligned}$$

(b)

$$\begin{aligned}d &= \int_0^1 |5\cos 2t|\,\mathrm{d}t\\ &= \int_0^{\frac{\pi}{4}} 5\cos(2t)\,\mathrm{d}t - \int_{\frac{\pi}{4}}^1 5\cos(2t)\,\mathrm{d}t\\ &= \left[\frac{5}{2}\sin 2t\right]_0^{\frac{\pi}{4}} - \left[\frac{5}{2}\sin 2t\right]_{\frac{\pi}{4}}^1\\ &= (2.5-0) - (2.5\sin 2 - 2.5)\\ &= 5 - 2.5\sin 2\\ &\approx 2.73\text{m}\end{aligned}$$

20. To prove:

$$\sin^3\theta = \frac{3\sin\theta - \sin(3\theta)}{4}$$

Proof:

$$\begin{aligned}\sin(3\theta) &= \mathrm{Im}(\mathrm{cis}(3\theta))\\ &= \mathrm{Im}(\mathrm{cis}^3\,\theta)\\ &= \mathrm{Im}(\cos^3\theta + 3\mathrm{i}\sin\theta\cos^2\theta\\ &\qquad - 3\sin^2\theta\cos\theta - \mathrm{i}\sin^3\theta)\\ &= 3\sin\theta\cos^2\theta - \sin^3\theta\\ &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta\\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\\ \text{RHS} &= \frac{3\sin\theta - \sin(3\theta)}{4}\\ &= \frac{3\sin\theta - (3\sin\theta - 4\sin^3\theta)}{4}\\ &= \frac{4\sin^3\theta}{4}\\ &= \sin^3\theta\\ &= \text{LHS}\end{aligned}$$

□

$$\begin{aligned}A &= \int_0^{\frac{\pi}{2}} |\sin^3\theta|\,\mathrm{d}\theta\\ &= \int_0^{\frac{\pi}{2}} \sin^3\theta\,\mathrm{d}\theta\\ &= \int_0^{\frac{\pi}{2}} \frac{3\sin\theta - \sin(3\theta)}{4}\,\mathrm{d}\theta\\ &= \left[\frac{-3\cos\theta + \frac{1}{3}\cos(3\theta)}{4}\right]_0^{\frac{\pi}{2}}\\ &= \left[\frac{\cos(3\theta) - 9\cos\theta}{12}\right]_0^{\frac{\pi}{2}}\\ &= \left(\frac{\cos\frac{3\pi}{2} - 9\cos\frac{\pi}{2}}{12}\right)\\ &\qquad - \left(\frac{\cos 0 - 9\cos 0}{12}\right)\\ &= 0 - \left(\frac{-8}{12}\right)\\ &= \frac{2}{3}\ \text{units}^2\end{aligned}$$

$$\begin{aligned}
A &= \int_0^{\frac{\pi}{2}} \sin^3\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \sin\theta(1-\cos^2\theta)\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} (\sin\theta - \cos^2\theta\sin\theta)\,\mathrm{d}\theta \\
&= \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}} \\
&= \left(-\cos\frac{\pi}{2} + \frac{\cos^3\frac{\pi}{2}}{3}\right) \\
&\quad - \left(-\cos 0 + \frac{\cos^3 0}{3}\right) \\
&= 0 - \left(-1+\frac{1}{3}\right) \\
&= \frac{2}{3}\text{ units}^2
\end{aligned}$$

21. (a)

		To (next location) A	B	C	D
From (present location)	A	0	1/2	0	1/2
	B	1/3	0	1/3	1/3
	C	0	1/2	0	1/2
	D	1/3	1/3	1/3	0

(b) 17 seconds is after 3 transitions, so

$$\begin{bmatrix} 0 & 0 & 200 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^3 = \begin{bmatrix} 22 & 78 & 22 & 78 \end{bmatrix}$$

(c) $$\begin{bmatrix} 0 & 0 & 0 & 200 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^3 = \begin{bmatrix} 52 & 52 & 52 & 44 \end{bmatrix}$$

(d) Intuitively I would expect 50% more people at B and D than at A and C, so about 40 at each of A and C and 60 at each of B and D. This would be my intuitive guess regardless of initial positions.

$$\begin{bmatrix} 200 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{15} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

check that this has in fact reached the long term average:

$$\begin{bmatrix} 200 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{16} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

(e) $$\begin{bmatrix} 0 & 200 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{15} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 200 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{16} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

(f) $$\begin{bmatrix} 50 & 50 & 50 & 50 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{15} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

$$\begin{bmatrix} 50 & 50 & 50 & 50 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{16} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

22.
$$\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} &= kT \\
\int \frac{\mathrm{d}T}{T} &= \int k\,\mathrm{d}t \\
\ln T &= kt + c \\
T &= T_0\mathrm{e}^{kt} \\
27.7 - 22 &= (28.6-22)\mathrm{e}^{1k} \\
k &= \ln\frac{27.7-22}{28.6-22} \\
&\approx -0.147 \\
28.6 - 22 &= (37-22)\mathrm{e}^{-0.147t} \\
-0.147t &= \ln\frac{28.6-22}{37-22} \\
t &= \frac{\ln\frac{28.6-22}{37-22}}{-0.147} \\
&= 5.6
\end{aligned}$$

5.6 hours before 1.30pm is 7:54am (although it's unlikely that such precision is realistic—8am is a more sensible estimate.)