Chapter 8

Exercise 8A

1. No working required.

2. (a) $k^2 = 4$ so $k = 2$ and $T = \frac{2\pi}{2} = \pi$.
   (b) $k^2 = 1$ so $k = 1$ and $T = \frac{2\pi}{1} = 2\pi$.
   (c) $k^2 = 25$ so $k = 5$ and $T = \frac{2\pi}{5} = 0.4\pi$.

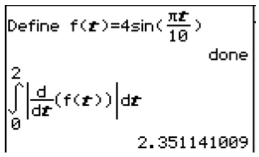
3. (a) $T = \frac{2\pi}{k} = 4\pi$ so $k = 0.5$ and $x = \sin 0.5t$.
   (b) $x = -\sin 0.5t$
   (c) $T = \frac{2\pi}{k} = \pi$ so $k = 2$ and $x = 3\sin 2t$.
   (d) $T = \frac{2\pi}{k} = 2$ so $k = \pi$ and $x = -0.5\sin \pi t$.

4. (a) $T = \frac{2\pi}{k} = \pi$ so $k = 2$ and $x = 2\cos 2t$.
   (b) $T = \frac{2\pi}{k} = 0.5\pi$ so $k = 4$ and $x = 1.5\cos 4t$.
   (c) $T = \frac{2\pi}{k} = 0.5$ so $k = 4\pi$ and $x = 0.5\cos 4\pi t$.

5. (a) $T = \frac{2\pi}{k} = \pi$ so $k = 2$ and $x = 2.5\sin 2t$ or $x = -2.5\sin 2t$ (depending on the direction of the motion at time $t = 0$).
   (b)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 5\cos 2t \\ &= 5\cos\frac{\pi}{3} \\ &= 2.5\mathrm{ms}^{-1} \end{aligned}$$

6. (a)
$$\begin{aligned} x &= 5\cos 5t + 3\sin 5t \\ &= r\sin(5t + \alpha) \end{aligned}$$
   where
$$\begin{aligned} r\sin\alpha &= 5 \\ r\cos\alpha &= 3 \\ \therefore \quad r^2 &= 5^2 + 3^2 \end{aligned}$$
   Hence the amplitude is $\sqrt{34}$m. (We could proceed to find the phase angle α but this is not requested by the question.)
   Period $T = \frac{2\pi}{5} = 0.4\pi$s.
   (b) Amplitude is $\sqrt{3^2 + 7^2} = \sqrt{58}$m. (You can use the approach in part (a) above, but you should probably remember the general result for questions like this.)
   Period $T = \frac{2\pi}{2} = \pi$s.

7. (a) To prove: $\ddot{x} = -k^2 x$
   Proof:
$$\begin{aligned} \dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t} 4\sin\frac{\pi t}{10} \\ &= \frac{2\pi}{5}\cos\frac{\pi t}{10} \\ \ddot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{2\pi}{5}\cos\frac{\pi t}{10} \\ &= -\frac{\pi^2}{25}\sin\frac{\pi t}{10} \\ &= -\frac{\pi^2}{10^2}\left(4\sin\frac{\pi t}{10}\right) \\ &= -\left(\frac{\pi}{10}\right)^2 x \end{aligned}$$
   Taking $k = \frac{\pi}{10}$ this gives $\ddot{x} = -k^2 x$. $\square$
   (In my opinion the wording of this question is a little unclear. Since it might well be reasonable to define simple harmonic motion as $x = a\sin(kt + \phi)$, the proof could be so trivial as to be non-existant. In order to proceed, I have taken the question to mean that we are required to prove that the motion satisfies the differential equation definition of SHM.)
   (b) The period of the motion is $T = 2\pi \times \frac{10}{\pi} = 20$s.
   Amplitude is 4m.
   (c) In the first two seconds the movement is all in the same direction and the distance moved is
$$\begin{aligned} d &= x(2) - x(0) \\ &= 4\sin\frac{\pi}{5} \\ &\approx 2.35\mathrm{m} \end{aligned}$$
   If using technology, it's probably simpler to take a definite integral of the absolute value of the velocity over the given interval:



   With this approach there is no need to first analyse whether the object changes direction during the interval under consideration: the absolute value takes care of that. Be warned, however, that handheld technology may take longer than a few seconds to evaluate this.

8. (a) To prove: $\ddot{x} = -k^2 x$
   Proof:
$$\begin{aligned} \dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t} 2\sin\frac{\pi t}{3} \\ &= \frac{2\pi}{3}\cos\frac{\pi t}{3} \\ \ddot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{2\pi}{3}\cos\frac{\pi t}{3} \\ &= -\frac{2\pi^2}{9}\sin\frac{\pi t}{3} \\ &= -\frac{\pi^2}{3^2}\left(2\sin\frac{\pi t}{3}\right) \\ &= -\left(\frac{\pi}{3}\right)^2 x \end{aligned}$$
   Taking $k = \frac{\pi}{3}$ this gives $\ddot{x} = -k^2 x$. $\square$

(b) The period of the motion is $T = 2\pi \times \frac{3}{\pi} = 6$s.

Amplitude is 2m.

(c) In the first two seconds the movement is not all in the same direction so the distance moved must be determined in two parts.

From $t = 0$ to $t = \frac{T}{4} = 1.5$ seconds the body moves through its amplitude: 2m.

From $t = 1.5$ to $t = 2$ seconds the body moves back to $x = 2\sin\frac{2\pi}{3} = \sqrt{3}$, thus moving through a further distance of $2 - \sqrt{3}$m.

Hence the total distance moved is

$$\begin{aligned} d &= (2) + (2 - \sqrt{3}) \\ &= (4 - \sqrt{3})\,\text{m} \end{aligned}$$

(See the note to the previous question about using technology.)

9. (a) To prove: $\ddot{x} = -k^2 x$

Proof:

$$\begin{aligned} \dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t} 3\sin\left(2t + \frac{\pi}{6}\right) \\ &= 6\cos\left(2t + \frac{\pi}{6}\right) \\ \ddot{x} &= \frac{\mathrm{d}}{\mathrm{d}t} 6\cos\left(2t + \frac{\pi}{6}\right) \\ &= -12\sin\left(2t + \frac{\pi}{6}\right) \\ &= -4\left(3\sin\left(2t + \frac{\pi}{6}\right)\right) \\ &= -2^2 x \end{aligned}$$

Taking $k = 2$ this gives $\ddot{x} = -k^2 x$. $\square$

(b) The period of the motion is $T = \frac{2\pi}{2} = \pi$s.

Amplitude is 3m.

(c) The body first reaches maximum displacement when

$$\begin{aligned} 3\sin\left(2t + \frac{\pi}{6}\right) &= 3 \\ 2t + \frac{\pi}{6} &= \frac{\pi}{2} \\ 2t &= \frac{\pi}{3} \\ t &= \frac{\pi}{6} \end{aligned}$$

From $t = 0$ to $t = \frac{\pi}{6}$ seconds the body moves

$$\begin{aligned} d &= 3 - 3\sin\left(2(0) + \frac{\pi}{6}\right) \\ &= 1.5\text{m} \end{aligned}$$

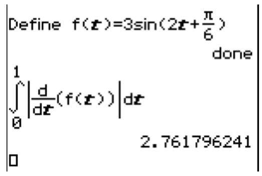
From $t = \frac{\pi}{6}$ to $t = 1$ seconds the body moves a further

$$\begin{aligned} d &= 3 - 3\sin\left(2(1) + \frac{\pi}{6}\right) \\ &\approx 1.26\text{m (2d.p.)} \end{aligned}$$

Hence the total distance moved is

$$\begin{aligned} d &= 1.5 + 1.26 \\ &= 2.76\text{m (2d.p.)} \end{aligned}$$

If using technology:



10. (a)

$$\begin{aligned} x &= a\sin(kt + \alpha) \\ a &= 4 \\ \frac{2\pi}{k} &= 2 \\ k &= \pi \\ 2 &= 4\sin(\pi(0) + \alpha) \\ \sin\alpha &= \frac{1}{2} \\ \alpha &= \frac{\pi}{6} \\ \text{or}\quad \alpha &= \frac{5\pi}{6} \\ v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 4\pi\cos(\pi t + \alpha) \end{aligned}$$

When $t = 0$,

$$\begin{aligned} v &= 4\pi\cos\alpha \\ \therefore\quad \alpha &= \frac{5\pi}{6} \text{ (to make } v \text{ negative)} \\ x &= 4\sin\left(\pi t + \frac{5\pi}{6}\right) \end{aligned}$$

(b)

$$\begin{aligned} v &= 4\pi\cos\left(\pi t + \frac{5\pi}{6}\right) \\ &= 4\pi\cos\left(\frac{\pi}{6} + \frac{5\pi}{6}\right) \\ &= 4\pi\cos\pi \\ &= -4\pi\,\text{ms}^{-1} \\ \therefore\ \text{speed} &= 4\pi\,\text{ms}^{-1} \end{aligned}$$

11. (a)

$$\begin{aligned}
x &= a\sin(kt+\alpha)\\
a &= 2\\
\frac{2\pi}{k} &= \frac{2\pi}{5}\\
k &= 5\\
\sqrt{2} &= 2\sin(5(0)+\alpha)\\
\sin\alpha &= \frac{\sqrt{2}}{2}\\
\alpha &= \frac{\pi}{4}\\
\text{or}\quad \alpha &= \frac{3\pi}{4}\\
v &= \frac{\mathrm{d}x}{\mathrm{d}t}\\
&= 10\cos(5t+\alpha)
\end{aligned}$$

When $t=0$,

$$\begin{aligned}
v &= 10\cos\alpha\\
\therefore\quad \alpha &= \frac{\pi}{4}\ \text{(to make } v \text{ positive)}\\
x &= 2\sin\left(5t+\frac{\pi}{4}\right)
\end{aligned}$$

(b) $v = 10\cos\left(5t+\frac{\pi}{4}\right)$ which has an amplitude of 10ms^{-1} so the greatest speed is 10ms^{-1}.

(c)

$$\begin{aligned}
a &= \frac{\mathrm{d}v}{\mathrm{d}t}\\
&= -50\sin\left(5t+\frac{\pi}{4}\right)
\end{aligned}$$

This has an amplitude of 50ms^{-2} so the maximum accelleration is 50ms^{-2}.

12.

$$\begin{aligned}
\ddot{x} &= -k^2x\\
&= -4x\\
\therefore\quad k &= 2\\
x &= 0.6\sin 2t
\end{aligned}$$

(a)

$$\begin{aligned}
x &= 0.6\sin\frac{2\pi}{6}\\
&= 0.6\sin\frac{\pi}{3}\\
&= 0.3\sqrt{3}\,\text{m}
\end{aligned}$$

(b)

$$\begin{aligned}
x &= 0.6\sin\frac{2\pi}{3}\\
&= -0.3\sqrt{3}\,\text{m}
\end{aligned}$$

(c)

$$\begin{aligned}
|x| &= 0.3\\
x &= \pm 0.3\\
0.6\sin 2t &= \pm 0.3\\
\sin 2t &= \pm 0.5\\
2t &\in \left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \dots\right\}\\
t &\in \left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \dots\right\}
\end{aligned}$$

i. $t = \frac{\pi}{12}$s
ii. $t = \frac{5\pi}{12}$s
iii. $t = \frac{7\pi}{12}$s

13.

$$\begin{aligned}
\ddot{x} &= -k^2x\\
&= -\pi^2x\\
\therefore\quad k &= \pi\\
x &= -3\sin\pi t
\end{aligned}$$

(a)

$$\begin{aligned}
x &= -3\sin\frac{\pi}{3}\\
&= -\frac{3\sqrt{3}}{2}\,\text{m}
\end{aligned}$$

(b)

$$\begin{aligned}
v &= \frac{\mathrm{d}x}{\mathrm{d}t}\\
&= -3\pi\cos\pi t\\
\text{when } t &= \frac{1}{3},\\
v &= -3\pi\cos\frac{\pi}{3}\\
&= -\frac{3\pi}{2}\,\text{ms}^{-1}
\end{aligned}$$

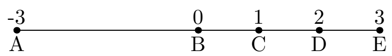
(c) speed$= |v| = \frac{3\pi}{2}\,\text{ms}^{-1}$

(d)

$$\begin{aligned}
|v| &= \frac{3\pi}{2}\\
v &= \pm\frac{3\pi}{2}\\
-3\pi\cos\pi t &= \pm\frac{3\pi}{2}\\
\cos\pi t &= \pm 0.5\\
\pi t &\in \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \dots\right\}\\
t &\in \left\{\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \dots\right\}
\end{aligned}$$

The body next has the same speed it had at $t=\frac{1}{3}$s when $t=\frac{2}{3}$s.

14.



Without loss of generality, suppose that the particle is at point A when $t=0$. Then

$$\begin{aligned}
x &= -3\cos kt\\
\frac{2\pi}{k} &= \pi\\
\therefore\quad k &= 2\\
\therefore\quad x &= -3\cos 2t
\end{aligned}$$

(a)

$$\begin{aligned}
x &= 1\\
-3\cos 2t &= 1\\
\cos 2t &= -\frac{1}{3}\\
t &= \frac{\cos^{-1} - \frac{1}{3}}{2}\\
&= 0.9553\dots\\
&\approx 0.96\text{s}
\end{aligned}$$

(b)
$$\begin{aligned} x &= 2 \\ -3\cos 2t &= 2 \\ \cos 2t &= -\frac{2}{3} \\ t &= \frac{\cos^{-1} -\frac{2}{3}}{2} \\ &= 1.1503\ldots \\ 1.1503 - 0.9553 &= 0.1949 \\ &\approx 0.19\text{s} \end{aligned}$$

(c)
$$\begin{aligned} x &= 3 \\ -3\cos 2t &= 3 \\ \cos 2t &= -1 \\ t &= \frac{\pi}{2} \\ &= 1.5708\ldots \\ 1.5708 - 1.1503 &= 0.4205 \\ &\approx 0.42\text{s} \end{aligned}$$

Check: the total times from A to E should be half the period: $\frac{\pi}{2} \approx 1.57$s–

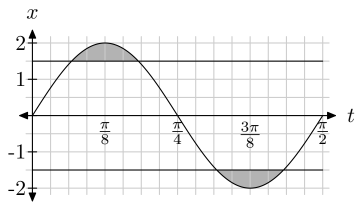
$$0.96 + 0.19 + 0.42 = 1.57$$

(d) If the particle is moving left-to-right when it passes D, the time to get back to D is double the time needed to go from D to E (since it moves from D to E and back again, and the symmetry makes these times DE and ED equal):

$$\begin{aligned} t &= 2 \times 0.4205 \\ &= 0.84\text{s} \end{aligned}$$

If the particle is moving right-to-left when it passes D, the time to get back to D again is a whole period less the D-E-D time, i.e.

$$\begin{aligned} t &= \pi - 0.84 \\ &\approx 2.30\text{s} \end{aligned}$$

15. 

Find the time the body first reaches 1.5m away from the mean position O and determine the length of time between that point and when it reaches maximum displacement. Then use the symmetry of the sine curve to determine the total time.

$$\begin{aligned} 2\sin 4t &= 1.5 \\ \sin 4t &= 0.75 \\ 4t &= \sin^{-1} 0.75 \\ t &= \frac{\sin^{-1} 0.75}{4} \\ &\approx 0.212 \\ \frac{\pi}{8} - t &\approx 0.181 \\ 0.181 \times 4 &\approx 0.72\text{s} \end{aligned}$$

16.
$$\begin{aligned} \ddot{x} &= -k^2 x \\ \ddot{x} &= -4x \\ k &= 2 \\ x &= a\sin(2t + \alpha) \\ v &= \dot{x} \\ &= 2a\cos(2t + \alpha) \end{aligned}$$

(a)
$$\begin{aligned} x &= a\sin(2t + \alpha) \\ 0 &= a\sin\alpha \\ \alpha &= 0 \\ v &= 2a\cos(2t + \alpha) \\ 4 &= 2a\cos 0 \\ a &= 2 \\ \therefore \quad x &= 2\sin 2t \end{aligned}$$

(b)
$$\begin{aligned} v &= 2a\cos(2t + \alpha) \\ 0 &= 2a\cos\alpha \\ \alpha &= \frac{\pi}{2} \\ x &= a\sin(2t + \alpha) \\ 4 &= a\sin\frac{\pi}{2} \\ a &= 4 \\ \therefore \quad x &= 4\sin\left(2t + \frac{\pi}{2}\right) \\ &= 4\cos 2t \end{aligned}$$

17. (a) Since the mass is at rest 2cm below equilibrium the amplitude of its motion is 2cm.

(b)
$$\begin{aligned} k^2 &= 64 \\ k &= 8 \\ \text{period} &= \frac{2\pi}{k} \\ &= \frac{\pi}{4}\text{ s} \end{aligned}$$

(c) This represents a quarter of a full cycle and takes a quarter of the period, i.e. $\frac{\pi}{16}$ s.

(d) The mass is at maximum speed when passing through the equilibrium point, so its speed is

$$\begin{aligned} s &= |ka| \\ &= 8 \times 2 \\ &= 16\text{cm/s} \end{aligned}$$

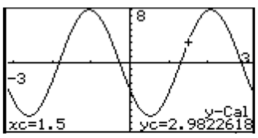
(e)
$$\begin{aligned} x &= -2\cos 8t \\ v &= 16\sin 8t \\ 16\sin 8t &= 8 \\ \sin 8t &= \frac{1}{2} \\ 8t &= \frac{\pi}{6} \\ t &= \frac{\pi}{48}\text{ s} \end{aligned}$$

18. (a)
$$\begin{aligned} x &= -4\sqrt{3}\sin 0 - 4\cos 0 \\ &= -4 \end{aligned}$$
The object is 4m from O.

(b) To prove:
$$\ddot{x} = -k^2 x$$
Proof:
$$\begin{aligned} x &= -4\sqrt{3}\sin 2t - 4\cos 2t \\ \dot{x} &= -8\sqrt{3}\cos 2t + 8\sin 2t \\ \ddot{x} &= 16\sqrt{3}\sin 2t + 16\cos 2t \\ &= -4(-4\sqrt{3}\sin 2t - 4\cos 2t) \\ &= -2^2 x \end{aligned}$$
as required, for $k = 2$. □

(c) This question is trivial to do using technology, as illustrated in previous questions. This gives an answer of
$$\int_0^{1.5} \left|\frac{\mathrm{d}x}{\mathrm{d}t}\right| dt = 14.98\text{m}$$
To work this question without technology is not within the scope of the course since it requires (amongst other things) calculating $\sin 3$ and $\cos 3$ which you are not expected to do without a calculator.

Another approach using technology is to graph the motion.



The object starts at $x = -4$, moves to the maximum negative displacement of -8 then comes back to $x = 2.98$ at $t = 1.5$. Thus the distance is $|-8 - -4| + |2.98 - -8| = 14.98$.

19. (a)
$$\begin{aligned} p &= x - 3 \\ &= 4\sin \pi t \\ \dot{p} &= 4\pi\cos \pi t \\ \ddot{p} &= -4\pi^2 \sin \pi t \\ &= -\pi^2 p \end{aligned}$$

(b) Period$= \frac{2\pi}{\pi} = 2$s.
Amplitude$= 4$

(c) The mean value of sine is zero, so the mean position of $3 + 4\sin \pi t$ is 3m.

(d) The maximum value of x occurs when $\sin \pi t = 1$:
$$x = 3 + 4 \times 1 = 7\text{m}$$

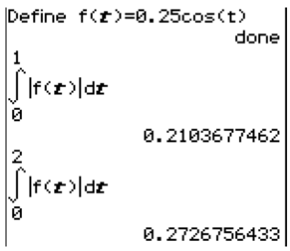
20. (a)
$$\begin{aligned} s &= x - 5 \\ &= -3\cos 2t \\ \dot{s} &= 6\sin 2t \\ \ddot{s} &= 12\cos 2t \\ &= -2^2 s \end{aligned}$$

(b) Period$= \frac{2\pi}{2} = \pi$s.
Amplitude$= 3$

(c) The mean value of cosine is zero, so the mean position of $5 - 3\cos 2t$ is 5m.

(d) The minimum value of x occurs when $\sin 2t = 1$:
$$x = 5 - 3 \times 1 = 2\text{m}$$

21. Determine distance by integrating speed—the absolute value of velocity.



22.
$$\begin{aligned} x &= a\sin(kt + \alpha) \\ v &= ka\cos(kt + \alpha) \end{aligned}$$
When $x = 20$, $v = 30$:
$$\begin{aligned} a\sin(kt+\alpha) &= 20 \\ \sin(kt+\alpha) &= \frac{20}{a} \\ ka\cos(kt+\alpha) &= 30 \\ \cos(kt+\alpha) &= \frac{30}{ka} \\ \sin^2(kt+\alpha) + \cos^2(kt+\alpha) &= 1 \\ \frac{400}{a^2} + \frac{900}{k^2a^2} &= 1 \\ 400k^2 + 900 &= k^2a^2 \end{aligned}$$

When $x = 24$, $v = 14$:

$$\begin{aligned}
a\sin(kt+\alpha) &= 24\\
\sin(kt+\alpha) &= \frac{24}{a}\\
ka\cos(kt+\alpha) &= 14\\
\cos(kt+\alpha) &= \frac{14}{ka}\\
\sin^2(kt+\alpha)+\cos^2(kt+\alpha) &= 1\\
\frac{576}{a^2}+\frac{196}{k^2a^2} &= 1\\
576k^2+196 &= k^2a^2\\
\text{and}\quad 400k^2+900 &= k^2a^2\\
\therefore\quad 576k^2+196 &= 400k^2+900\\
176k^2 &= 704\\
k^2 &= 4\\
k &= \pm 2\\
\text{Period} &= \frac{2\pi}{2}\\
&= \pi\,\text{s}\\
\text{Now}\quad 400k^2+900 &= k^2a^2\\
1600+900 &= 4a^2\\
a^2 &= 625\\
a &= \pm 25\\
\text{Amplitude} &= 25\,\text{m}
\end{aligned}$$

23.
$$\begin{aligned}
x &= a\sin(kt+\alpha)\\
v &= ka\cos(kt+\alpha)
\end{aligned}$$

When $x = 0.6$, $v = 0.75$:

$$\begin{aligned}
a\sin(kt+\alpha) &= 0.6\\
\sin(kt+\alpha) &= \frac{0.6}{a}\\
ka\cos(kt+\alpha) &= 0.76\\
\cos(kt+\alpha) &= \frac{0.75}{ka}\\
\sin^2(kt+\alpha)+\cos^2(kt+\alpha) &= 1\\
\frac{0.36}{a^2}+\frac{0.5625}{k^2a^2} &= 1\\
0.36k^2+0.5625 &= k^2a^2
\end{aligned}$$

When $x = 0.39$, $v = 1.56$:

$$\begin{aligned}
a\sin(kt+\alpha) &= 0.39\\
\sin(kt+\alpha) &= \frac{0.39}{a}\\
ka\cos(kt+\alpha) &= 1.56\\
\cos(kt+\alpha) &= \frac{1.56}{ka}
\end{aligned}$$

$$\begin{aligned}
\sin^2(kt+\alpha)+\cos^2(kt+\alpha) &= 1\\
\frac{0.1521}{a^2}+\frac{2.4336}{k^2a^2} &= 1\\
0.1521k^2+2.4336 &= k^2a^2\\
\text{and}0.36k^2+0.5625 &= k^2a^2\\
\therefore\quad 0.1521k^2+2.4336 &= 0.36k^2+0.5625\\
0.2079k^2 &= 1.8711\\
k^2 &= 9\\
k &= \pm 3\\
\text{Period} &= \frac{2\pi}{3}\,\text{s}\\
\text{Now}\quad 0.36k^2+0.5625 &= k^2a^2\\
3.24+0.5625 &= 9a^2\\
a^2 &= 0.4225\\
a &= \pm 0.65\\
\text{Amplitude} &= 65\,\text{cm}
\end{aligned}$$

24. (a) To prove: $\ddot{x} = -k^2x$

Proof:

$$\begin{aligned}
x &= A\cos kt\\
\dot{x} &= -kA\sin kt\\
\ddot{x} &= -k^2A\cos kt\\
&= -k^2x
\end{aligned}$$

as required. □

To prove: $|x| \le |A\cos 0|$

Proof:

$$\begin{aligned}
\text{RHS} &= |A|\\
\text{LHS} &= |A\cos kt|\\
&= |A||\cos kt|\\
|\cos kt| &\le 1\\
\therefore\quad |A||\cos kt| &\le |A|\\
\text{LHS} &\le \text{RHS}
\end{aligned}$$

as required. □

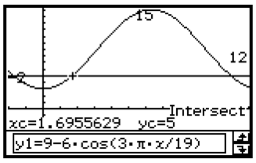
(b) Modelling the tide movement with SHM gives a mean depth of $\frac{3+15}{2} = 9$m and an amplitude of $\frac{15-3}{2} = 6$m. The period is double the time between low and high tides, i.e. $6\frac{1}{3}\times 2 = 12\frac{2}{3}$ hours. Hence

$$\begin{aligned}
\frac{2\pi}{k} &= 12\frac{2}{3}\\
&= \frac{38}{3}\\
38k &= 6\pi\\
k &= \frac{3\pi}{19}
\end{aligned}$$

If we take 7am as our starting time (i.e. $t = 0$ at 7am) the water depth is

$$d = 9 - 6\cos\frac{3\pi t}{19}$$

To determine the times when the water depth is at least 5m, plot a graph of this function and determine when it exceeds 5:



This gives $1.70 \le t \le 10.97$ representing times (to the nearest 5 minutes) between 8:40am and 6:00pm.

Integration By Parts Extension Exercise

1.
$$u = x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \qquad v = -\cos x$$

$$\begin{aligned}\int x\sin x\,\mathrm{d}x &= x(-\cos x) - \int -\cos x\,\mathrm{d}x\\ &= -x\cos x + \sin x + c\\ &= \sin x - x\cos x + c\end{aligned}$$

2.
$$u = x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x \qquad v = \sin x$$

$$\begin{aligned}\int x\cos x\,\mathrm{d}x &= x\sin x - \int \sin x\,\mathrm{d}x\\ &= x\sin x - \cos x + c\end{aligned}$$

3.
$$u = 3x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x \qquad v = -0.5\cos 2x$$

$$\begin{aligned}&\int 3x\sin 2x\,\mathrm{d}x\\ &= 3x(-0.5\cos 2x) - \int (-0.5\cos 2x)(3)\,\mathrm{d}x\\ &= -1.5x\cos 2x + 0.75\sin 2x + c\\ &= \frac{3\sin x - 6x\cos 2x}{4} + c\end{aligned}$$

4.
$$u = x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \qquad v = 0.5\mathrm{e}^{2x}$$

$$\begin{aligned}\int x\mathrm{e}^{2x}\,\mathrm{d}x &= x(0.5\mathrm{e}^{2x}) - \int 0.5\mathrm{e}^{2x}\,\mathrm{d}x\\ &= 0.5x\mathrm{e}^{2x} - 0.25\mathrm{e}^{2x}\\ &= \frac{(2x-1)\mathrm{e}^{2x}}{4} + c\end{aligned}$$

5.
$$u = \ln x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^2 \qquad v = \frac{x^3}{3}$$

$$\begin{aligned}\int x^2\ln x\,\mathrm{d}x &= \frac{x^3\ln x}{3} - \int \frac{x^3}{3x}\,\mathrm{d}x\\ &= \frac{x^3\ln x}{3} - \int \frac{x^2}{3}\,\mathrm{d}x\\ &= \frac{x^3\ln x}{3} - \frac{x^3}{9} + c\\ &= \frac{x^3(3\ln x - 1)}{9} + c\end{aligned}$$

6.
$$u = x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = (x+2)^5 \qquad v = \frac{(x+2)^6}{6}$$

$$\begin{aligned}\int x(x+2)^5\,\mathrm{d}x &= \frac{x(x+2)^6}{6} - \int \frac{(x+2)^6}{6}\,\mathrm{d}x\\ &= \frac{x(x+2)^6}{6} - \frac{(x+2)^7}{42} + c\\ &= \frac{(x+2)^6(7x-(x+2))}{42} + c\\ &= \frac{(x+2)^6(6x-2)}{42} + c\\ &= \frac{(x+2)^6(3x-1)}{21} + c\end{aligned}$$

7.
$$u = x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{2x+1} \qquad v = \frac{2(2x+1)^{\frac{3}{2}}}{3 \times 2} = \frac{(2x+1)^{\frac{3}{2}}}{3}$$

$$\begin{aligned}
\int x\sqrt{2x+1}\,\mathrm{d}x &= \frac{x(2x+1)^{\frac{3}{2}}}{3} - \int \frac{(2x+1)^{\frac{3}{2}}}{3}\,\mathrm{d}x \\
&= \frac{x(2x+1)^{\frac{3}{2}}}{3} - \frac{2(2x+1)^{\frac{5}{2}}}{15 \times 2} + c \\
&= \frac{5x(2x+1)^{\frac{3}{2}} - (2x+1)^{\frac{5}{2}}}{15} + c \\
&= \frac{(2x+1)^{\frac{3}{2}}(5x-(2x+1)}{15} + c \\
&= \frac{(2x+1)^{\frac{3}{2}}(5x-2x-1)}{15} + c \\
&= \frac{(2x+1)^{\frac{3}{2}}(3x-1)}{15} + c
\end{aligned}$$

8.
$$u = x^2 \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \qquad v = \mathrm{e}^x$$

$$\int x^2\mathrm{e}^x\,\mathrm{d}x = x^2\mathrm{e}^x - \int 2x\mathrm{e}^x\,\mathrm{d}x$$

The integral on the right hand side requires integration by parts again.

$$u = 2x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \qquad v = \mathrm{e}^x$$

$$\begin{aligned}
\int 2x\mathrm{e}^x\,\mathrm{d}x &= 2x\mathrm{e}^x - \int 2\mathrm{e}^x\,\mathrm{d}x \\
&= 2x\mathrm{e}^x - 2\mathrm{e}^x + c \\
&= 2\mathrm{e}^x(x-1) + c \\
\therefore \int x^2\mathrm{e}^x\,\mathrm{d}x &= x^2\mathrm{e}^x - 2\mathrm{e}^x(x-1) + c \\
&= \mathrm{e}^x(x^2 - 2x + 2) + c
\end{aligned}$$

9.
$$u = x^2 \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \qquad v = -\cos x$$

$$\begin{aligned}
\int x^2\sin x\,\mathrm{d}x &= -x^2\cos x - \int -2x\cos x\,\mathrm{d}x \\
&= \int 2x\cos x\,\mathrm{d}x - x^2\cos x
\end{aligned}$$

The integral on the right hand side requires integration by parts again.

$$u = 2x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x \qquad v = \sin x$$

$$\begin{aligned}
\int 2x\cos x\,\mathrm{d}x &= 2x\sin x - \int 2\sin x\,\mathrm{d}x \\
&= 2x\sin x + 2\cos x + c \\
\therefore \int x^2\sin x\,\mathrm{d}x &= 2x\sin x + 2\cos x - x^2\cos x + c
\end{aligned}$$

10. The key to this problem is appropriate selection of u and v so that differentiating one and integrating the other leaves an expression that is more amenable to integration. Often this means that we need to end up with a lower power of x. Differentiation of e^{x^2} will not achieve this, so we need to look to integrate this part of the expression.

$$u = x^2 \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = 2x\mathrm{e}^{x^2} \qquad v = \mathrm{e}^{x^2}$$

$$\begin{aligned}
\int x^3\mathrm{e}^{x^2}\,\mathrm{d}x &= \int (x^2)(2x\mathrm{e}^{x^2})\,\mathrm{d}x \\
&= x^2\mathrm{e}^{x^2} - \int 2x\mathrm{e}^{x^2}\,\mathrm{d}x \\
&= x^2\mathrm{e}^{x^2} - \mathrm{e}^{x^2} + c \\
&= \mathrm{e}^{x^2}(x^2-1) + c
\end{aligned}$$

11.
$$u = \ln x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \qquad v = x$$

$$\begin{aligned}
\int \ln x\,\mathrm{d}x &= x\ln x - \int \frac{x}{x}\,\mathrm{d}x \\
&= x\ln x - x + c \\
&= x(\ln x - 1) + c
\end{aligned}$$

12.
$$u = \sin x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \qquad v = \mathrm{e}^x$$

$$\int \mathrm{e}^x\sin x\,\mathrm{d}x = \mathrm{e}^x\sin x - \int \mathrm{e}^x\cos x\,\mathrm{d}x$$

The integral on the right needs to be done by parts again. Are we going around in circles? (You were warned these are sneaky!)

$$u = \cos x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \qquad v = \mathrm{e}^x$$

$$\int e^x \cos x = e^x \cos x + \int e^x \sin x \, dx$$
$$\therefore \int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$
$$\therefore 2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$
$$\int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2} + c$$

13.
$$u = \cos 2x \qquad \frac{du}{dx} = -2\sin 2x$$
$$\frac{dv}{dx} = e^x \qquad v = e^x$$

$$\int e^x \cos 2x \, dx = e^x \cos 2x - \int -2e^x \sin 2x \, dx$$
$$= e^x \cos 2x + 2\int e^x \sin 2x \, dx$$

The integral on the right needs to be done by parts again.

$$u = \sin 2x \qquad \frac{du}{dx} = 2\cos 2x$$
$$\frac{dv}{dx} = e^x \qquad v = e^x$$

$$\int e^x \sin 2x = e^x \sin 2x - \int 2e^x \cos 2x \, dx$$
$$= e^x \sin 2x - 2\int e^x \cos 2x \, dx$$
$$\therefore \int e^x \cos 2x \, dx = e^x \cos 2x + 2\Big(e^x \sin 2x - 2\int e^x \cos 2x \, dx\Big)$$
$$= e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x \, dx$$
$$\therefore 5\int e^x \cos 2x \, dx = e^x \cos 2x + 2e^x \sin 2x$$
$$\int e^x \cos 2x \, dx = \frac{e^x(\cos 2x + 2\sin 2x)}{5} + c$$

Miscellaneous Exercise 8

1. (c)
$$BA = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ 3 & 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0-6 & -2-4 & 1+0 \\ 0+0 & 2+0 & -1+0 \end{bmatrix}$$
$$= \begin{bmatrix} -6 & -6 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$
$$BA + C = \begin{bmatrix} -5 & -6 & 1 \\ -2 & 3 & 2 \end{bmatrix}$$

(f)
$$BD = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -1-4 \\ 1+0 \end{bmatrix}$$
$$= \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$
$$BD + D = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

2. (a) For $x < -6$, $|x+6| = -(x+6)$ and $|x-2| = -(x-2)$:

$$-(x+6) = 2 - (x-2)$$
$$-x - 6 = 2 - x + 2$$
$$-6 = 4 \Longrightarrow \text{no solution}$$

For $-6 \le x < 2$, $|x+6| = x+6$ and $|x-2| = -(x-2)$:

$$x + 6 = 2 - (x-2)$$
$$= 2 - x + 2$$
$$2x + 6 = 4$$
$$x = -1$$

For $x \ge 2$, $|x+6| = x+6$ and $|x-2| = x-2$:

$$x + 6 = 2 + x - 2$$
$$6 = 0 \Longrightarrow \text{no solution}$$

Single solution: $x = -1$.

(b) For $x < -6$, $|x+6| = -(x+6)$ and $|x-2| = -(x-2)$:

$$-(x+6) = 10 - (x-2)$$
$$-x - 6 = 10 - x + 2$$
$$-6 = 12 \Longrightarrow \text{no solution}$$

For $-6 \le x < 2$, $|x+6| = x+6$ and $|x-2| = -(x-2)$:

$$\begin{aligned} x+6 &= 10-(x-2)\\ &= 10-x+2\\ 2x+6 &= 12\\ x &= 3 \Longrightarrow \text{outside the domain} \end{aligned}$$

For $x \ge 2$, $|x+6| = x+6$ and $|x-2| = x-2$:

$$\begin{aligned} x+6 &= 10+x-2\\ 6 &= 8 \Longrightarrow \text{no solution} \end{aligned}$$

No solution.

(c) For $x < -6$, $|x+6| = -(x+6)$ and $|x-2| = -(x-2)$:

$$\begin{aligned} -(x+6) &= 8-(x-2)\\ -x-6 &= 8-x+2\\ -6 &= 10 \Longrightarrow \text{no solution} \end{aligned}$$

For $-6 \le x < 2$, $|x+6| = x+6$ and $|x-2| = -(x-2)$:

$$\begin{aligned} x+6 &= 8-(x-2)\\ &= 8-x+2\\ 2x+6 &= 10\\ x &= 2 \Longrightarrow \text{outside the domain} \end{aligned}$$

For $x \ge 2$, $|x+6| = x+6$ and $|x-2| = x-2$:

$$\begin{aligned} x+6 &= 8+x-2\\ 6 &= 6 \Longrightarrow \text{all solution} \end{aligned}$$

The solution is $x \ge 2$.

3.
$$\begin{bmatrix} 2x & 6\\ 4 & 2y \end{bmatrix} + \begin{bmatrix} 3y & 3\\ -3 & -3x \end{bmatrix} = \begin{bmatrix} 7 & 9\\ 1 & 22 \end{bmatrix}$$
$$\begin{bmatrix} 2x+3y & 9\\ 1 & 2y-3x \end{bmatrix} = \begin{bmatrix} 7 & 9\\ 1 & 22 \end{bmatrix}$$
$$\begin{aligned} \therefore \quad & 2x+3y = 7\\ \text{and} \quad & -3x+2y = 22 \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 2 & 3\\ -3 & 2 \end{bmatrix}\begin{bmatrix} x\\ y \end{bmatrix} &= \begin{bmatrix} 7\\ 22 \end{bmatrix}\\ \begin{bmatrix} x\\ y \end{bmatrix} &= \frac{1}{13}\begin{bmatrix} 2 & -3\\ 3 & 2 \end{bmatrix}\begin{bmatrix} 7\\ 22 \end{bmatrix}\\ &= \frac{1}{13}\begin{bmatrix} -52\\ 65 \end{bmatrix}\\ &= \begin{bmatrix} -4\\ 5 \end{bmatrix} \end{aligned}$$
$$\therefore \quad x = -4, \quad y = 5$$

(You could, of course, use non-matrix techniques to solve the simultaneous equations if you so choose.)

4.
$$\begin{aligned} PA &= P + 2A\\ PA - P &= 2A\\ P(A-I) &= 2A\\ P &= 2A(A-I)^{-1}\\ &= 2\begin{bmatrix} 2 & -1\\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1\\ 1 & 1 \end{bmatrix}^{-1}\\ &= 2\begin{bmatrix} 2 & -1\\ 1 & 2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1\\ -1 & 1 \end{bmatrix}\\ &= \begin{bmatrix} 3 & 1\\ -1 & 3 \end{bmatrix} \end{aligned}$$

5.
$$\begin{aligned} A^2 &= \begin{bmatrix} p & -p\\ 0 & q \end{bmatrix}\begin{bmatrix} p & -p\\ 0 & q \end{bmatrix}\\ &= \begin{bmatrix} p^2 & -p^2-pq\\ 0 & q^2 \end{bmatrix}\\ \det A &= pq\\ pqA^{-1} &= \begin{bmatrix} q & p\\ 0 & p \end{bmatrix}\\ p^2qA^{-1} &= \begin{bmatrix} pq & p^2\\ 0 & p^2 \end{bmatrix}\\ A^2 + p^2qA^{-1} &= \begin{bmatrix} p^2 & -p^2-pq\\ 0 & q^2 \end{bmatrix} + \begin{bmatrix} pq & p^2\\ 0 & p^2 \end{bmatrix}\\ &= \begin{bmatrix} p^2+pq & -pq\\ 0 & p^2+q^2 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} p^2+pq & -pq\\ 0 & p^2+q^2 \end{bmatrix} = \begin{bmatrix} 6 & 3\\ 0 & 10 \end{bmatrix}$$
$$\begin{aligned} pq &= -3\\ p^2+pq &= 6\\ p^2-3 &= 6\\ p^2 &= 9\\ p &= \pm 3\\ \therefore \quad q &= \mp 1\\ \text{check:} \quad p^2+q^2 &= 10\\ 9+1 &= 10 \quad \text{ok.}\\ \therefore \quad (p,q) &\in \{(3,-1),(-3,1)\} \end{aligned}$$

6. Let $A = (x,y)$ be some arbitrary point on the $x-y$ plane. The matrix $\begin{bmatrix} a & b\\ ka & kb \end{bmatrix}$ transforms this point to $A' = (x',y')$, thus:

$$\begin{aligned} \begin{bmatrix} x'\\ y' \end{bmatrix} &= \begin{bmatrix} a & b\\ ka & kb \end{bmatrix}\begin{bmatrix} x\\ y \end{bmatrix}\\ &= \begin{bmatrix} ax+by\\ kax+kby \end{bmatrix}\\ &= \begin{bmatrix} ax+by\\ k(ax+by) \end{bmatrix}\\ \text{thus} \quad x' &= ax+by\\ \text{and} \quad y' &= k(ax+by)\\ &= kx' \end{aligned}$$

Thus the transformed point satisfies the equation $y = kx$ and hence lies on the line as required. $\square$

7. (a)
$$\begin{aligned}\dot{x} &= 8\cos 4t\\ \ddot{x} &= -32\sin 4t\\ &= -4^2 x\end{aligned}$$
Period$=\frac{2\pi}{4}=\frac{\pi}{2}$s.
When $t=0$, $x=2\sin 0=0$.
The mean position is at O (distance=0).

(b)
$$\begin{aligned}\dot{x} &= -15\sin 3t\\ \ddot{x} &= -75\cos 3t\\ &= -3^2 x\end{aligned}$$
Period$=\frac{2\pi}{3}$s.
When $t=0$, $x=5\cos 0=5$m.
The mean position is at O (distance=0).

(c)
$$\begin{aligned}\dot{x} &= 4\cos 2t-8\sin 2t\\ \ddot{x} &= -8\sin 2t-16\cos 2t\\ &= -2^2 x\end{aligned}$$
Period$=\frac{2\pi}{2}=\pi$s.
When $t=0$, $x=2\cos 0+4\sin 0=2$m.
The mean position is at O (distance=0).

(d)
$$\begin{aligned}\dot{x} &= 15\cos 5t\\ \ddot{x} &= -75\sin 5t\\ &= -5^2(x-1)\end{aligned}$$
Period$=\frac{2\pi}{5}$s.
When $t=0$, $x=1+3\sin 0=1$m.
The mean position of $3\sin 5t$ is 0, so the mean position of $1+3\sin 5t$ is 1m from O.

8. The volume of any prism-like solid is equal to the area of the base times the height. The height here is 5m and the area of the base is determined by
$$\begin{aligned}A &= -\int_0^\pi -\sin x\,\mathrm{d}x\\ &= -\left[\cos x\right]_0^\pi\\ &= -(-1-1)\\ &= 2\text{m}^2\end{aligned}$$
Thus the volume of sand required is 10m^3.

9. First, rewrite each relation with x the dependent variable:
$$\begin{aligned}x &= y+3\\ x &= y^2+1\end{aligned}$$
Now find the points of intersection to determine the bounds for our integrals:
$$\begin{aligned}y+3 &= y^2+1\\ y^2-y-2 &= 0\\ (y-2)(y+1) &= 0\\ y &= -1\\ \text{and}\quad y &= 2\end{aligned}$$
The region we want is right of the parabola and left of the line, i.e. $y^2+1\le x\le y+3$, so the area is
$$\begin{aligned}A &= \int_{-1}^2 (y+3)-(y^2+1)\,\mathrm{d}y\\ &= \int_{-1}^2 (y-y^2+2)\,\mathrm{d}y\\ &= \left[\frac{y^2}{2}-\frac{y^3}{3}+2y\right]_{-1}^2\\ &= \left(\frac{4}{2}-\frac{8}{3}+4\right)-\left(\frac{1}{2}+\frac{1}{3}-2\right)\\ &= \frac{10}{3}-\left(-\frac{7}{6}\right)\\ &= \frac{27}{6}\\ &= 4.5\text{ units}^2\end{aligned}$$

10. (c) Let P_0 represent the initial population:
$$P_0=\begin{bmatrix}340\\720\\840\\220\\80\end{bmatrix}$$
i.
$$LP_0=\begin{bmatrix}1922\\204\\504\\672\\198\end{bmatrix}\approx\begin{bmatrix}1920\\200\\500\\670\\200\end{bmatrix}$$
ii.
$$L^{10}P_0=\begin{bmatrix}3836\\2213\\1404\\875\\738\end{bmatrix}\approx\begin{bmatrix}3800\\2200\\1400\\900\\700\end{bmatrix}$$

(d) i. $\begin{bmatrix}1&1&1&1&1\end{bmatrix}L^{19}P_0=[25\,775]$

ii. $\begin{bmatrix}1&1&1&1&1\end{bmatrix}L^{20}P_0=[28\,839]$

iii. $\begin{bmatrix}1&1&1&1&1\end{bmatrix}L^{29}P_0=[80\,618]$

iv. $\begin{bmatrix}1&1&1&1&1\end{bmatrix}L^{30}P_0=[90\,367]$

v. $\begin{bmatrix}1&1&1&1&1\end{bmatrix}L^{49}P_0=[791\,220]$

vi. $\begin{bmatrix}1&1&1&1&1\end{bmatrix}L^{50}P_0=[886\,936]$

(e)
$$\frac{P_{20}}{P_{19}} = 1.119$$
$$\frac{P_{30}}{P_{29}} = 1.121$$
$$\frac{P_{50}}{P_{49}} = 1.121$$
This suggests an annual growth rate of 12.1%.

(f)
$$\frac{1}{1.121} = 0.892$$
$$1 - 0.892 = 0.108$$
The harvesting rate should be 10.8%.

(g)
$$(0.95L)^5 P_0 = \begin{bmatrix} 2067 \\ 873 \\ 442 \\ 515 \\ 450 \end{bmatrix} \approx \begin{bmatrix} 2050 \\ 850 \\ 450 \\ 500 \\ 450 \end{bmatrix}$$

11. One approach is to use Euler's formula:

$$\begin{aligned}\text{L.H.S.} &= (\cos\theta + \mathrm{i}\sin\theta)^n \\ &= \left(\mathrm{e}^{\mathrm{i}\theta}\right)^n \\ &= \mathrm{e}^{\mathrm{i}n\theta} \\ &= \cos n\theta + \mathrm{i}\sin n\theta \\ &= \text{R.H.S.}\end{aligned}$$

□

12. This is a pretty standard kind of exam question.

$$\begin{aligned}\cos 4\theta + \mathrm{i}\sin 4\theta &= (\cos\theta + \mathrm{i}\sin\theta)^4 \\ &= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta \\ &\quad + 6\mathrm{i}^2\cos^2\theta\sin^2\theta + 4\mathrm{i}^3\cos\theta\sin^3\theta \\ &\quad + \mathrm{i}^4\sin^4\theta \\ &= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta \\ &\quad - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta \\ &\quad + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &\quad + \mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)\end{aligned}$$

Equating real parts,

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) \\ &\quad + (1-\cos^2\theta)^2 \\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta \\ &\quad + 1 - 2\cos^2\theta + \cos^4\theta \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

13. Since the particle has positive acceleration for all $t \geq 0$ and positive initial velocity, its velocity is always positive and the distance travelled in the third second is the difference between its position at $t = 2$ and at $t = 3$.

$$\begin{aligned}v(t) &= \int a(t)\,\mathrm{d}t \\ &= \int (6t+4)\,\mathrm{d}t \\ &= 3t^2 + 4t + c \\ x(t) &= \int v(t)\,\mathrm{d}t \\ &= \int (3t^2 + 4t + c)\,\mathrm{d}t \\ &= t^3 + 2t^2 + ct + k\end{aligned}$$

$$\begin{aligned}x(3) - x(2) &= 32 \\ (45 + 3c + k) - (16 + 2c + k) &= 32 \\ 29 + c &= 32 \\ c &= 3 \\ \therefore \quad v(1) &= 10\,\mathrm{ms}^{-1}\end{aligned}$$

14. (a) Let a be the surface area and s the side length.

$$\begin{aligned}a &= 6s^2 \\ \frac{\mathrm{d}a}{\mathrm{d}s} &= 12s \\ \frac{\delta a}{\delta s} &\approx 12s \\ \delta a &\approx 12s\delta s \\ &= 12 \times 5 \times 0.2 \\ &= 12\,\mathrm{cm}^2\end{aligned}$$

(b) Let v be the volume and s the side length.

$$\begin{aligned}v &= s^3 \\ \frac{\mathrm{d}v}{\mathrm{d}s} &= 3s^2 \\ \frac{\delta v}{\delta s} &\approx 3s^2 \\ \delta v &\approx 3s^2\delta s \\ &= 3 \times 25 \times 0.2 \\ &= 15\,\mathrm{cm}^3\end{aligned}$$

15. (a)
$$\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{200}{1+x}$$

(b)
$$\begin{aligned}\frac{200}{1+x} &= 2 \\ 1 + x &= 100 \\ x &= 99 \\ C &= 600 + 200\ln(1+99) \\ &= 1521.03 \\ \frac{C}{x} &= \frac{1521.03}{99} \\ &= \$15.36 \text{ per unit}\end{aligned}$$

16.
$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= -0.005A \\ A &= A_0\mathrm{e}^{-0.005t}\end{aligned}$$

The half-life represents the time when $A = 0.5A_0$:

$$\begin{aligned} 0.5A_0 &= A_0\mathrm{e}^{-0.005t} \\ \mathrm{e}^{-0.005t} &= 0.5 \\ -0.005t &= \ln 0.5 \\ &= -\ln 2 \\ t &= \frac{\ln 2}{0.005} \\ &\approx 139 \text{ years} \end{aligned}$$

17.
$$\begin{aligned} C &= C_0\mathrm{e}^{-kt} \\ 0.5C_0 &= C_0\mathrm{e}^{-5700k} \\ -5700k &= \ln 0.5 \\ &= -\ln 2 \\ k &= \frac{\ln 2}{5700} \\ \therefore \quad C &= C_0\mathrm{e}^{-\frac{t\ln 2}{5700}} \\ &= C_0(\mathrm{e}^{\ln 2})^{-\frac{t}{5700}} \\ &= C_0 2^{-\frac{t}{5700}} \end{aligned}$$

Given that 65% has decayed, $C = 0.35C_0$,

$$\begin{aligned} 0.35C_0 &= C_0 2^{-\frac{t}{5700}} \\ 2^{-\frac{t}{5700}} &= 0.35 \\ -\frac{t}{5700} &= \log_2 0.35 \\ t &= -5700\log_2 0.35 \\ &\approx 8\,600 \text{ years} \end{aligned}$$

18.
$$\begin{aligned} C &= C_0\mathrm{e}^{-kt} \\ 0.5C_0 &= C_0\mathrm{e}^{-12k} \\ -12k &= \ln 0.5 \\ &= -\ln 2 \\ k &= \frac{\ln 2}{12} \\ \therefore \quad C &= C_0\mathrm{e}^{-\frac{t\ln 2}{12}} \\ &= C_0(\mathrm{e}^{\ln 2})^{-\frac{t}{12}} \\ &= C_0 2^{-\frac{t}{12}} \end{aligned}$$

Given $C = 0.05C_0$,

$$\begin{aligned} 0.05C_0 &= C_0 2^{-\frac{t}{12}} \\ 2^{-\frac{t}{12}} &= 0.05 \\ -\frac{t}{12} &= \log_2 0.05 \\ t &= -12\log_2 0.05 \\ &\approx 52 \text{ days} \end{aligned}$$

19. The proposition to prove is:

$$5^n + 3\times 9^n = 4a, \quad a, n \in \mathbb{I}, n \geq 0$$

Proof:

For $n = 0$:

$$\begin{aligned} 5^0 + 3\times 9^0 &= 1 + 4 \\ &= 4 \end{aligned}$$

Assume the proposition is true for $n = k$, i.e.:

$$5^k + 3\times 9^k = 4a$$

for some integer a.

Then for $n = k+1$,

$$\begin{aligned} 5^{k+1} + 3{\times}9^{k+1} & \\ &= 5\times 5^k + 3\times 9\times 9^k \\ &= (4+1)\times 5^k + (8+1)\times 3\times 9^k \\ &= 4\times 5^k + 8\times 3\times 9^k + 5^k + 3\times 9^k \\ &= 4(\times 5^k + 2\times 3\times 9^k) + 4a \\ &= 4(\times 5^k + 2\times 3\times 9^k + a) \end{aligned}$$

Hence if the proposition is true for $n = k$ then it is also true for $n = k+1$, and since it is true for $n = 0$ it is true for all integer $n \geq 0$ by mathematical induction. $\square$

20. (a)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{9}{1+t} - 4 \\ v &= 0 \\ \frac{9}{1+t} - 4 &= 0 \\ \frac{9}{1+t} &= 4 \\ 9 &= 4(1+t) \\ 4 + 4t &= 9 \\ 4t &= 5 \\ t &= 1.25\mathrm{s} \end{aligned}$$

(b)
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= -\frac{9}{(1+t)^2} \\ v &= a \\ \frac{9}{1+t} - 4 &= -\frac{9}{(1+t)^2} \\ -9(1+t) + 4(1+t)^2 &= 9 \\ 4t^2 + 8t + 4 - 9t - 9 &= 9 \\ 4t^2 - t - 14 &= 0 \\ (4t+7)(t-2) &= 0 \\ t &= 2 \end{aligned}$$

(discarding the negative solution for t because we are given $t \geq 0$).

21. Repeatedly rotate 90° anti-clockwise to give $z_2 = -b + a\mathrm{i}$, $z_3 = -a - b\mathrm{i}$, $z_4 = b - a\mathrm{i}$.

22. (a)
$$\begin{aligned} A &= \int 6\mathrm{e}^{2t}\,\mathrm{d}t \\ &= 3\mathrm{e}^{2t} + c \\ 4 &= 3\mathrm{e}^0 + c \\ c &= 1 \\ A &= 3\mathrm{e}^{2t} + 1 \end{aligned}$$

(b)
$$\begin{aligned} A &= 3\mathrm{e}^1 + 1 \\ &= 3\mathrm{e} + 1 \end{aligned}$$

(c)
$$\begin{aligned} \delta A &\approx \frac{\mathrm{d}A}{\mathrm{d}t}\delta t \\ &= 6\mathrm{e}^0 \times 0.01 \\ &= 0.06 \end{aligned}$$

23. Starting from De Moivre's Theorem,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

Let $n = 2$

$$\begin{aligned} (\cos\theta + \mathrm{i}\sin\theta)^2 &= \cos 2\theta + \mathrm{i}\sin 2\theta \\ \cos^2\theta + 2\mathrm{i}\cos\theta\sin\theta + \mathrm{i}^2\sin^2\theta &= \cos 2\theta + \mathrm{i}\sin 2\theta \\ \cos^2\theta - \sin^2\theta + 2\mathrm{i}\cos\theta\sin\theta &= \cos 2\theta + \mathrm{i}\sin 2\theta \end{aligned}$$

Equating real parts gives

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

Equating imaginary parts gives

$$2\cos\theta\sin\theta = \sin 2\theta$$

as required. □

24. (a) If you recognise this as being of the form $\int (f(x))^n f'(x)\,\mathrm{d}x$ where $f(x) = \ln x$ then this can be done by inspection: no working required.

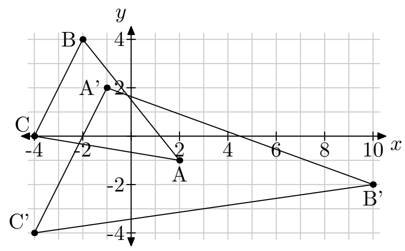
(b) This can also be done by inspection.

(c)
$$\begin{aligned} \int \frac{(\ln x^4)}{x}\,\mathrm{d}x &= \int \frac{4\ln x}{x}\,\mathrm{d}x \\ &= 2(\ln x)^2 + c \end{aligned}$$

25. (a)
$$\begin{aligned} T^{-1} &= \frac{1}{3}\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix} \\ A &= T^{-1}A' \\ &= \frac{1}{3}\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ B &= T^{-1}B' \\ &= \frac{1}{3}\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 10 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ C &= T^{-1}C' \\ &= \frac{1}{3}\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ 0 \end{bmatrix} \end{aligned}$$

The coordinates of A, B and C are $(2, -1)$, $(-2, 4)$ and $(-4, 0)$ respectively.

(b) $|\det T| = 3$



Let $|\triangle ABC|$ represent the area of triangle ABC. We can determine the area of each triangle by considering its enclosing rectangle and subtracting the right-triangular regions outside the triangle.

$$\begin{aligned} |\triangle ABC| &= 6 \times 5 \\ &\quad - \frac{2\times 4}{2} - \frac{4\times 5}{2} - \frac{6\times 1}{2} \\ &= 13\text{units}^2 \\ |\triangle A'B'C'| &= 14 \times 6 \\ &\quad - \frac{3\times 6}{2} - \frac{11\times 4}{2} - \frac{14\times 2}{2} \\ &= 39\text{units}^2 \\ |\triangle A'B'C'| &= 3|\triangle ABC| \end{aligned}$$

26. (a)
$$\begin{aligned} v(t) &= \int 0.1\mathrm{e}^{0.1t}\,\mathrm{d}t \\ &= \mathrm{e}^{0.1t} + c \\ v(0) &= 0 \\ \mathrm{e}^0 + c &= 0 \\ c &= -1 \\ v(t) &= \mathrm{e}^{0.1t} - 1 \\ v(10) &= (\mathrm{e} - 1)\,\mathrm{ms}^{-1} \end{aligned}$$

(b)
$$\begin{aligned}
x(t) &= \int (\mathrm{e}^{0.1t} - 1)\,\mathrm{d}t \\
&= 10\mathrm{e}^{0.1t} - t + c \\
x(0) &= 0 \\
10\mathrm{e}^0 - 0 + c &= 0 \\
c &= -10 \\
x(t) &= 10\mathrm{e}^{0.1t} - t - 10 \\
x(10) &= (10\mathrm{e} - 20)\,\mathrm{m}
\end{aligned}$$

(c) Distance travelled is equal to the difference in displacement, provided there is no change in sign in velocity. $v = \mathrm{e}^{0.1t} - 1$ is positive for all $t > 0$ so

$$\begin{aligned}
d(T) &= x(T+1) - x(T) \\
&= 10\mathrm{e}^{0.1(T+1)} - (T+1) - 10 \\
&\quad - (10\mathrm{e}^{0.1T} - T - 10) \\
&= 10\mathrm{e}^{0.1(T+1)} - 1 - 10\mathrm{e}^{0.1T} \\
&= 10\mathrm{e}^{0.1}\mathrm{e}^{0.1T} - 10\mathrm{e}^{0.1T} - 1 \\
&= (10\mathrm{e}^{0.1T}(\mathrm{e}^{0.1} - 1) - 1)\,\mathrm{m}
\end{aligned}$$

(d) The third second means from $t = 2$ to $t = 3$, so we want $d(2)$:

$$\begin{aligned}
d(2) &= 10\mathrm{e}^{0.2}(\mathrm{e}^{0.1} - 1) - 1 \\
&= 0.285\,\mathrm{m}
\end{aligned}$$

(e)
$$\begin{aligned}
d(9) &= 10\mathrm{e}^{0.9}(\mathrm{e}^{0.1} - 1) - 1 \\
&= 1.587\,\mathrm{m}
\end{aligned}$$

27. Although this presents itself as a transition matrix question, it can be answered more intuitively. The long-term distribution will be that distribution that results in a steady state, i.e. when the 6% of the birds at A who switch to B are balanced by the 4% of the birds at B who switch to B.

Let a be the number of birds at A.

Let b be the number of birds at B.

$$\begin{aligned}
0.06a &= 0.04b \\
1.5a &= b \\
\frac{a}{a+b} &= \frac{a}{a+1.5a} \\
&= \frac{1}{2.5} \\
&= 0.4
\end{aligned}$$

Forty percent of the birds will be at A in the long term.

Here is the matrix approach:

$$T = \begin{array}{cc} & \begin{array}{cc} & \textit{From} \\ & \begin{array}{cc} A & B \end{array} \end{array} \\ \text{To} & \begin{array}{c} A \\ B \end{array} \begin{bmatrix} 0.94 & 0.04 \\ 0.06 & 0.96 \end{bmatrix} \end{array}$$

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{aligned}
0.94a + 0.04b &= a \\
0.06a + 0.96b &= b \\
-0.06a + 0.04b &= 0 \\
0.06a - 0.04b &= 0 \\
b &= 1.5a \\
\frac{a}{a+b} &= \frac{a}{a+1.5a} \\
&= \frac{1}{2.5} \\
&= 0.4
\end{aligned}$$

Alternatively, if using technology, once you've formed T, simply raise it to increasingly high powers until the two columns are sufficiently identical and interpret the results.

28. (a)
$$\begin{aligned}
\int \sin^3 x\,\mathrm{d}x &= \int \sin^2 x \sin x\,\mathrm{d}x \\
&= \int (1 - \cos^2 x)\sin x\,\mathrm{d}x \\
&= \int (\sin x - \cos^2 x \sin x)\,\mathrm{d}x \\
&= -\cos x + \frac{\cos^3 x}{3} + c
\end{aligned}$$

(b)
$$\begin{aligned}
\int 4\sin^2 x\,\mathrm{d}x &= \int -2(-2\sin^2 x)\,\mathrm{d}x \\
&= -2\int (1 - 2\sin^2 x - 1)\,\mathrm{d}x \\
&= -2\int (\cos 2x - 1)\,\mathrm{d}x \\
&= -2\left(\frac{\sin 2x}{2} - x\right) + c \\
&= 2x - \sin 2x + c
\end{aligned}$$

29. (a) No working required.

(b)
$$\begin{aligned}
u &= 4x \\
\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{4x} \mathrm{e}^{t^2}\,\mathrm{d}t &= \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}u}\int_1^u \mathrm{e}^{t^2}\,\mathrm{d}t \\
&= 4\mathrm{e}^{u^2} \\
&= 4\mathrm{e}^{(4x)^2} \\
&= 4\mathrm{e}^{16x^2}
\end{aligned}$$

30.
$$\begin{aligned}
M &= M_0\mathrm{e}^{-kt}\\
0.5M_0 &= M_0\mathrm{e}^{-30k}\\
-30k &= \ln 0.5\\
&= -\ln 2\\
k &= \frac{\ln 2}{30}\\
\therefore\quad M &= M_0\mathrm{e}^{-\frac{t\ln 2}{30}}\\
&= M_0(\mathrm{e}^{\ln 2})^{-\frac{t}{30}}\\
&= M_0 2^{-\frac{t}{30}}
\end{aligned}$$

Given M_0 is 20 times the safe level, we need $M = 0.05M_0$,

$$\begin{aligned}
0.05M_0 &= M_0 2^{-\frac{t}{30}}\\
2^{-\frac{t}{30}} &= 0.05\\
-\frac{t}{30} &= \log_2 0.05\\
t &= -30\log_2 0.05\\
&\approx 130 \text{ years}
\end{aligned}$$

31. (a) There are two paths from A: to D with probability 0.7 and to B with probability p. Since the probabilities must add to 1, $p = 0.3$. Similarly there are two paths from C: to D with probability 0.4 and to B with probability q, giving $q = 0.6$.

(b) Let T be the transition matrix as follows:

$$T = \begin{array}{cc} & \textit{From} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{To}\ \begin{array}{c} A\\B\\C\\D \end{array} & \begin{bmatrix} 0 & 0.4 & 0 & 0.4\\ 0.3 & 0 & 0.6 & 0.5\\ 0 & 0.1 & 0 & 0.1\\ 0.7 & 0.5 & 0.4 & 0 \end{bmatrix} \end{array}$$

Let S_0 be the initial state matrix:

$$S_0 = \begin{bmatrix} 1000\\0\\0\\0 \end{bmatrix}$$

After one period,

$$\begin{aligned}
S_1 &= TS_0\\
&= \begin{bmatrix} 0\\300\\0\\700 \end{bmatrix}
\end{aligned}$$

That is, 300 people at B and 700 at D.

(c)
$$\begin{aligned}
S_2 &= T^2S_0\\
&= \begin{bmatrix} 400\\350\\100\\150 \end{bmatrix}
\end{aligned}$$

That is, 400 people at A, 350 at B, 100 at C and 150 at D.

(d)
$$\begin{aligned}
S_3 &= T^3S_0\\
&= \begin{bmatrix} 200\\255\\50\\495 \end{bmatrix}
\end{aligned}$$

That is, 200 people at A, 255 at B, 50 at C and 495 at D.

(e)
$$T^{20}S_0 = \begin{bmatrix} 267\\302\\67\\364 \end{bmatrix}$$

$$T^{21}S_0 = \begin{bmatrix} 267\\302\\67\\364 \end{bmatrix}$$

In the long term there are expected to be 267 people at A, 302 at B, 67 at C and 364 at D.

Alternatively, solve

$$\begin{aligned}
T\begin{bmatrix} a\\b\\c\\d \end{bmatrix} &= \begin{bmatrix} a\\b\\c\\d \end{bmatrix}\\
T\begin{bmatrix} a\\b\\c\\d \end{bmatrix} - \begin{bmatrix} a\\b\\c\\d \end{bmatrix} &= \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}\\
(T-I)\begin{bmatrix} a\\b\\c\\d \end{bmatrix} &= \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}\\
-a + 0.4b + 0.4d &= 0\\
0.3a - b + 0.6c + 0.5d &= 0\\
0.1b - c + 0.1d &= 0\\
0.7a + 0.5b + 0.4c - d &= 0\\
\text{and } a + b + c + d &= 1000
\end{aligned}$$

(Actually one of the first four of these five equations is redundant and can be left out when solving.)

This gives us

$$\begin{aligned}
a &= \frac{800}{3}\\
b &= \frac{2720}{9}\\
c &= \frac{200}{3}\\
d &= \frac{3280}{9}
\end{aligned}$$

Although you should be able to solve four equations in four unknowns, it's difficult to envisage a situation where you would need to do that without the assistance of appropriate technology. With that in mind, the simpler first approach is probably more appropriate.

32. There are several conjectures that could be made here. This solution addresses only the most obvious.

The initial conjecture might deal with only two, three and four digit numbers:

Conjecture: The difference between a 2-, 3- or 4-digit natural number and its reflection is a multiple of 9 (where a number's reflection is another number with the same digits in reverse order).

The obvious extension is to consider numbers with more digits.

Conjecture: The difference between any natural number and its reflection is a multiple of 9.

Test with some five digit numbers:

$$|12345 - 54321| = 41976 = 9 \times 4664$$
$$|10209 - 90201| = 79992 = 9 \times 8888$$
$$|92654 - 45629| = 47025 = 9 \times 5225$$

These support the conjecture.

Proof: For two digit numbers, let a and b be single-digit natural numbers. Any two digit natural number p can be represented as

$$p = 10a + b$$

and the reflection of p as

$$p^R = 10b + a$$

This gives the difference as

$$\begin{aligned} |p - p^R| &= |10a + b - (10b + a)| \\ &= |9a - 9b| \\ &= 9|a - b| \end{aligned}$$

Thus the difference is 9 times the difference between the two digits: a multiple of 9 as required.

For three digit numbers, let a, b and c be single digit natural numbers. Any three digit natural number p can be represented as

$$p = 100a + 10b + c$$

and the reflection of p as

$$p^R = 100c + 10b + a$$

This gives the difference as

$$\begin{aligned} |p - p^R| &= |100a + 10b + c - (100c + 10b + a)| \\ &= |99a - 99c| \\ &= 99|a - c| \end{aligned}$$

Thus the difference is 99 times the difference between the first and last digits: a multiple of 9 as required.

(A four-digit proof could be given next, but let's be more ambitious.)

Now consider the conjecture for n-digit natural numbers. Assume the conjecture to be true for any number p with k digits, that is

$$p - p^R = 9d$$

for some integer d.

Let a and b be single digit natural numbers. If we put a before the digits of p and put b after, we create a new natural number q having $k + 2$ digits:

$$q = 10^{k+1}a + 10p + b$$

and the reflection of q is

$$q^R = 10^{k+1}b + 10p^R + a$$

Then

$$\begin{aligned} &|q - q^R| \\ &= |10^{k+1}a + 10p + b - (10^{k+1}b + 10p^R + a)| \\ &= |(10^{k+1} - 1)a + 10(p - p^R) - (10^{k+1} - 1)b| \\ &= |(10^{k+1} - 1)(a - b) + 90d| \end{aligned}$$

One less than any positive power of 10 is a multiple of 9 (which we could also prove by induction, but we take as self-evident here) so we can conclude that if the conjecture is true for numbers having k digits then it is also true for numbers having $k+2$ digits. Since we have established the conjecture for numbers having 2 and 3 digits, it is proven for all numbers of 2 or more digits by mathematical induction. □

33.
$$\begin{aligned} \left(\frac{z_1}{z_2 z_3}\right)^{-3} &= \left(\frac{\sqrt{6}\operatorname{cis}\frac{5\pi}{6}}{(2\operatorname{cis}\frac{\pi}{2})(3\operatorname{cis}\frac{2\pi}{3})}\right)^{-3} \\ &= \left(\frac{\sqrt{6}\operatorname{cis}\frac{5\pi}{6}}{6\operatorname{cis}\frac{7\pi}{6}}\right)^{-3} \\ &= \left(\frac{\operatorname{cis}-\frac{\pi}{3}}{\sqrt{6}}\right)^{-3} \\ &= 6\sqrt{6}\operatorname{cis}\pi \\ &= -6\sqrt{6} \end{aligned}$$

34. (a) No working required.

(b)
$$\begin{aligned} z^{14} &= (\sqrt{2}\operatorname{cis}-\frac{\pi}{4})^{14} \\ &= 2^{\frac{14}{2}}\operatorname{cis}-\frac{14\pi}{4} \\ &= 2^7\operatorname{cis}-\frac{7\pi}{2} \\ &= 128\operatorname{cis}\frac{\pi}{2} \\ &= 128\text{i} \end{aligned}$$

35.
$$\begin{aligned}
z &= -1+\sqrt{3}\mathrm{i}\\
&= \sqrt{(-1)^2+(\sqrt{3})^2}\operatorname{cis}\tan^{-1}\frac{\sqrt{3}}{-1}\\
&\quad (2^{\text{nd}}\text{ quadrant})\\
&= 2\operatorname{cis}\frac{2\pi}{3}\\
\bar{z} &= 2\operatorname{cis}-\frac{2\pi}{3}\\
\frac{1}{\bar{z}} &= \frac{1}{2}\operatorname{cis}\frac{2\pi}{3}\\
\left(z+\frac{1}{\bar{z}}\right)^4 &= \left(2\operatorname{cis}\frac{2\pi}{3}+\frac{1}{2}\operatorname{cis}\frac{2\pi}{3}\right)^4\\
&= \left(\frac{5}{2}\operatorname{cis}\frac{2\pi}{3}\right)^4\\
&= \frac{625}{16}\operatorname{cis}\frac{8\pi}{3}\\
&= \frac{625}{16}\operatorname{cis}\frac{2\pi}{3}\\
\left(z-\frac{1}{\bar{z}}\right)^4 &= \left(2\operatorname{cis}\frac{2\pi}{3}-\frac{1}{2}\operatorname{cis}\frac{2\pi}{3}\right)^4\\
&= \left(\frac{3}{2}\operatorname{cis}\frac{2\pi}{3}\right)^4\\
&= \frac{81}{16}\operatorname{cis}\frac{8\pi}{3}\\
&= \frac{81}{16}\operatorname{cis}\frac{2\pi}{3}
\end{aligned}$$

36. Let d be the distance AB.

$$\begin{aligned}
\sin\theta &= \frac{h}{50}\\
\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{1}{50}\frac{\mathrm{d}h}{\mathrm{d}t}\\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{1.2}{50\cos\theta}\\
d^2+h^2 &= 2500\\
2d\frac{\mathrm{d}d}{\mathrm{d}t}+2h\frac{\mathrm{d}h}{\mathrm{d}t} &= 0\\
d\frac{\mathrm{d}d}{\mathrm{d}t}+h\frac{\mathrm{d}h}{\mathrm{d}t} &= 0\\
d\frac{\mathrm{d}d}{\mathrm{d}t}+1.2h &= 0\\
\frac{\mathrm{d}d}{\mathrm{d}t} &= -\frac{1.2h}{d}
\end{aligned}$$

When $h = 40$,

$$\begin{aligned}
d &= \sqrt{2500-1600}\\
&= 30\,\mathrm{m}\\
\cos\theta &= \frac{30}{50}\\
&= 0.6\\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{1.2}{50\cos\theta}\\
&= \frac{1.2}{30}\\
&= 0.04\text{ radians per second}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}d}{\mathrm{d}t} &= -\frac{1.2h}{d}\\
&= -\frac{1.2\times 40}{30}\\
&= -1.6\,\mathrm{ms}^{-1}
\end{aligned}$$

B approaches A at 1.6 metres per second.

37. (a) $12\times 5\,000+5\times 8\,000=\$100\,000$

(b)
$$\begin{aligned}
AD &= 12-DC\\
&= 12-\frac{5}{\tan\theta}\\
DB &= \frac{5}{\sin\theta}\\
C &= 5\,000AD+8\,000DB\\
&= 5\,000\left(12-\frac{5}{\tan\theta}\right)+8\,000\left(\frac{5}{\sin\theta}\right)\\
&= 60\,000-\frac{25\,000}{\tan\theta}+\frac{40\,000}{\sin\theta}
\end{aligned}$$

as required.

(c) Minimum cost will be at one or other extremes of the domain, or where $\frac{\mathrm{d}C}{\mathrm{d}\theta}=0$. Extremes are where D is coincident with point C—with cost of \$100 000 as seen in part (a)—or where D is coincident with point A, in which case the cost is

$$8\,000\sqrt{5^2+12^2}=\$104\,000$$

$$\begin{aligned}
\frac{\mathrm{d}C}{\mathrm{d}\theta} &= \frac{25\,000}{\tan^2\theta\cos^2\theta}-\frac{40\,000\cos\theta}{\sin^2\theta}\\
&= \frac{25\,000}{\sin^2\theta}-\frac{40\,000\cos\theta}{\sin^2\theta}
\end{aligned}$$

Setting $\frac{\mathrm{d}C}{\mathrm{d}\theta}=0$ gives

$$\begin{aligned}
\frac{25\,000}{\sin^2\theta}-\frac{40\,000\cos\theta}{\sin^2\theta} &= 0\\
25\,000-40\,000\cos\theta &= 0\\
\cos\theta &= \frac{25\,000}{40\,000}\\
&= 0.625\\
\theta &= 51^\circ
\end{aligned}$$

$$\begin{aligned} C &= 60\,000 - \frac{25\,000}{\tan 51^\circ} + \frac{40\,000}{\sin 51^\circ} \\ &= \$91\,000 \end{aligned}$$

38. For particle A, the amplitude of the displacement gives $c = 5$.

    The velocity for particle A is

    $$v = k_1 c \cos k_1 t$$

    and from the graph,

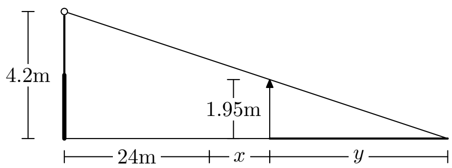
    $$\begin{aligned} k_1 c &= 10 \\ k_1 &= 2 \\ \text{Period} &= \frac{2\pi}{k_1} \\ &= \pi \,\text{s} \end{aligned}$$

    For particle B,

    $$\begin{aligned} v &= k_2 d \cos k_2 t \\ a &= -k_2^2 d \sin k_2 t \\ k_2 d &= 3 \\ k_2^2 d &= 1.5 \\ k_2 &= 0.5 \\ d &= 6 \\ \text{Period} &= \frac{2\pi}{k_2} \\ &= 4\pi \,\text{s} \end{aligned}$$

    Note: the answer of $k_1 = 1$ in Sadler is an error.

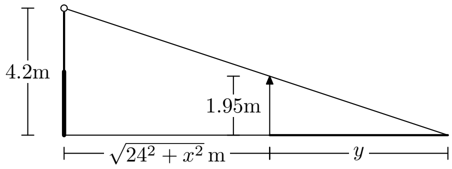
39. Let y be the length of the shadow and let x be the distance that has been run.

    (a) 

    $$\begin{aligned} \frac{y}{24 + x + y} &= \frac{1.95}{4.2} \\ 4.2y &= 1.95(24 + x + y) \\ 2.25y &= 1.95x + 46.8 \\ y &= \frac{13}{15}x + \frac{937}{45} \\ \frac{dy}{dx} &= \frac{13}{15} \\ \frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} \\ &= \frac{13}{15} \times 5 \\ &= \frac{13}{3} \text{ m/s} \end{aligned}$$

    (b) The geometry is exactly the same as in (a) except that the sign of x is reversed. The length of the shadow changes at the same speed, but now it is getting shorter at $\frac{13}{3}$ m/s instead of getting longer.

    (c) After the runner has travelled x metres, the distance from the lamppost is given by Pythagoras' Theorem:

    $$\begin{aligned} \frac{y}{\sqrt{24^2 + x^2} + y} &= \frac{1.95}{4.2} \\ 4.2y &= 1.95(\sqrt{24^2 + x^2} + y) \\ 2.25y &= 1.95\sqrt{24^2 + x^2} \\ y &= \frac{13\sqrt{24^2 + x^2}}{15} \\ \frac{dy}{dx} &= \frac{13x}{15\sqrt{24^2 + x^2}} \\ \frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} \\ &= \frac{13x}{15\sqrt{24^2 + x^2}} \times 5 \\ &= \frac{13x}{3\sqrt{24^2 + x^2}} \end{aligned}$$

    When $t = 2$, $x = 10$

    $$\begin{aligned} \frac{dy}{dt} &= \frac{130}{3\sqrt{24^2 + 10^2}} \\ &= \frac{5}{3} \text{ m/s} \end{aligned}$$

40. (a) Let U be the amount used in millions of tonnes, then

    $$\begin{aligned} R &= 5e^{0.08t} \\ U &= \int_0^{10} R \, dt \\ &= \int_0^{10} 5e^{0.08t} \, dt \\ &= \left[\frac{5e^{0.08t}}{0.08}\right]_0^{10} \\ &= \left[62.5e^{0.08t}\right]_0^{10} \\ &= 62.5(e^{0.8} - e^0) \\ &= 76.60 \text{ million tonnes} \end{aligned}$$

    (b)
    $$\begin{aligned} \frac{dA}{dt} &= -5e^{0.08t} \\ A(t) &= \int -5e^{0.08t} \, dt \\ &= -62.5e^{0.08t} + c \\ A(0) &= 200 \\ 200 &= -62.5e^0 + c \\ c &= 262.5 \\ A(t) &= 262.5 - 62.5e^{0.08t} \end{aligned}$$

Solving for t when $A(t) = 0$,

$$\begin{aligned}262.5 - 62.5\mathrm{e}^{0.08t} &= 0\\ 62.5\mathrm{e}^{0.08t} &= 262.5\\ \mathrm{e}^{0.08t} &= 4.2\\ 0.08t &= \ln(4.2)\\ t &= \frac{\ln(4.2)}{0.08}\\ &\approx 17.9\end{aligned}$$

The resource will be exhausted before the end of the eighteenth year.

41. (a) $x = \sin u \qquad \mathrm{d}x = \cos u\,\mathrm{d}u$
$u = \sin^{-1} x$

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\,\mathrm{d}u\\ &= \int \frac{\cos u}{\sqrt{\cos^2 u}}\,\mathrm{d}u\\ &= \int \frac{\cos u}{|\cos u|}\,\mathrm{d}u\\ &= u + c\\ &= \sin^{-1} x + c\end{aligned}$$

Note that it is safe to disregard the absolute value since we can cover all permissable values of x by restricting u to first or fourth quadrants where $\cos u \geq 0$.

(b) $x = 5\sin u \qquad \mathrm{d}x = 5\cos u\,\mathrm{d}u$
$u = \sin^{-1} \frac{x}{5}$

$$\begin{aligned}\int \frac{1}{\sqrt{25-x^2}}\,\mathrm{d}x &= \int \frac{5\cos u}{\sqrt{25-5^2\sin^2 u}}\,\mathrm{d}u\\ &= \int \frac{5\cos u}{\sqrt{25\cos^2 u}}\,\mathrm{d}u\\ &= \int \frac{5\cos u}{|5\cos u|}\,\mathrm{d}u\\ &= u + c\\ &= \sin^{-1} \frac{x}{5} + c\end{aligned}$$

(c) $x = \frac{3}{2}\sin u \qquad \mathrm{d}x = \frac{3}{2}\cos u\,\mathrm{d}u$
$u = \sin^{-1} \frac{2x}{3}$

$$\begin{aligned}\int \frac{1}{\sqrt{9-4x^2}}\,\mathrm{d}x &= \int \frac{\frac{3}{2}\cos u}{\sqrt{9-4\left(\frac{3}{2}\right)^2\sin^2 u}}\,\mathrm{d}u\\ &= \int \frac{\frac{3}{2}\cos u}{\sqrt{9\cos^2 u}}\,\mathrm{d}u\\ &= \int \frac{\frac{3}{2}\cos u}{|3\cos u|}\,\mathrm{d}u\\ &= \frac{u}{2} + c\\ &= \frac{1}{2}\sin^{-1} \frac{2x}{3} + c\end{aligned}$$

(d) $x = \sin u \qquad \mathrm{d}x = \cos u\,\mathrm{d}u$
$u = \sin^{-1} x$

$$\begin{aligned}\int \sqrt{1-x^2}\,\mathrm{d}x &= \int \sqrt{1-\sin^2 u}\cos u\,\mathrm{d}u\\ &= \int \sqrt{\cos^2 u}\cos u\,\mathrm{d}u\\ &= \int \cos^2 u\,\mathrm{d}u\\ &= \frac{1}{2}\int (2\cos^2 u - 1 + 1)\,\mathrm{d}u\\ &= \frac{1}{2}\int (\cos 2u + 1)\,\mathrm{d}u\\ &= \frac{\sin 2u}{4} + \frac{u}{2} + c\\ &= \frac{2\sin u\cos u}{4} + \frac{u}{2} + c\\ &= \frac{\sin u\sqrt{1-\sin^2 u}}{2} + \frac{u}{2} + c\\ &= \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2} + c\end{aligned}$$

(e) $x = 2\sin u \qquad \mathrm{d}x = 2\cos u\,\mathrm{d}u$
$u = \sin^{-1} \frac{x}{2}$

$$\begin{aligned}&\int \sqrt{4-x^2}\,\mathrm{d}x\\ &= \int \sqrt{4-4\sin^2 u}(2\cos u)\,\mathrm{d}u\\ &= \int 2\sqrt{4\cos^2 u}\cos u\,\mathrm{d}u\\ &= \int 4\cos^2 u\,\mathrm{d}u\\ &= 2\int (2\cos^2 u - 1 + 1)\,\mathrm{d}u\\ &= 2\int (\cos 2u + 1)\,\mathrm{d}u\\ &= \sin 2u + 2u + c\\ &= 2\sin u\cos u + 2u + c\\ &= 2\sin u\sqrt{1-\sin^2 u} + 2u + c\\ &= x\sqrt{1-\frac{x^2}{4}} + 2\sin^{-1}\frac{x}{2} + c\\ &= \frac{x\sqrt{4-x^2}}{2} + 2\sin^{-1}\frac{x}{2} + c\end{aligned}$$

(f) $x = 2\cos u \qquad \mathrm{d}x = -2\sin u\,\mathrm{d}u$

$u = \cos^{-1}\dfrac{x}{2}$

$$\begin{aligned}
\int \sqrt{4-x^2}\,\mathrm{d}x &\\
&= \int \sqrt{4-4\cos^2 u}(-2\sin u)\,\mathrm{d}u\\
&= \int -2\sqrt{4\sin^2 u}\sin u\,\mathrm{d}u\\
&= \int -4\sin^2 u\,\mathrm{d}u\\
&= 2\int (1-2\sin^2 u-1)\,\mathrm{d}u\\
&= 2\int (\cos 2u-1)\,\mathrm{d}u\\
&= \sin 2u - 2u + c\\
&= 2\sin u\cos u - 2u + c\\
&= 2\sqrt{1-\cos^2 u}\cos -2u + c\\
&= x\sqrt{1-\frac{x^2}{4}} - 2\cos^{-1}\frac{x}{2} + c\\
&= \frac{x\sqrt{4-x^2}}{2} - 2\cos^{-1}\frac{x}{2} + c
\end{aligned}$$

(Comparing (e) and (f) might suggest that

$$\sin^{-1} x = -\cos^{-1} x$$

but this is not the case because the constants of integration in these two answers are different.)